Eigenvalue asymptotics for randomly perturbed non-selfadjoint operators

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Abstract

We consider quite general h-pseudodifferential operators on $\mathbf{R}^n$ with small random perturbations and show that in the limit $h \to 0$ the eigenvalues are distributed according to a Weyl law with a probabality that tends to 1. The first author has previously obtained a similar result in dimension 1. Our class of perturbations is different.

Résumé

Nous considérons des opérateurs h-pseudodifférentiels assez généraux et nous montrons que dans la limite $h \to 0$, les valeurs propres se distribuent selon une loi de Weyl, avec une probabilité qui tend vers 1. Le premier auteur a déjà obtenu un résultat semblable en dimension 1. Notre classe de perturbations est différente.

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Contents

1 Introduction

This work can be viewed as a continuation of [7], where one of us studied random perturbations of non-selfadjoint h-pseudodifferential operators on $\mathbf{R}$ and showed that Weyl asymptotics holds with a probability that is very close to 1. In the present work we consider the multidimensional case and weaken some of the assumptions in [7] (like independence of the differentials and analyticity of the symbol). Our random perturbations are slighly different however, in [7] they are given by a random potential while here they are rather given by a random integral operator.

Before continuing the general discussion, we fix the framework more in detail. We will work in the semi-classical limit on $\mathbf{R}^n$. Write $\rho = (x,\xi)$ and let $m \geq 1$ be an order function on the phase space $\mathbf{R}^{2n}_{x,\xi}$:

$$\begin{gathered}\exists C_0 \geq 1,\ N_0 > 0 \text{ such that } m(\rho) \leq C_0 \langle \rho - \mu \rangle^{N_0} m(\mu), \\ \forall \rho,\mu \in \mathbf{R}^{2n},\ \langle \rho-\mu\rangle = \sqrt{1+|\rho-\mu|^2}.\end{gathered} \tag{1.1}$$

The corresponding symbol space (cf [2]) is then

$$S(\mathbf{R}^{2n},m) = \{a \in C^\infty(\mathbf{R}^{2n});\ |\partial_\rho^\alpha a(\rho)| \leq C_\alpha m(\rho),\ \rho\in\mathbf{R}^{2n},\ \alpha\in\mathbf{N}^{2n}\}. \tag{1.2}$$

Let

$$P(\rho; h) \sim p(\rho) + hp_1(\rho) + ... \text{ in } S(\mathbf{R}^{2n}, m). \tag{1.3}$$

Assume $\exists\, z_0 \in \mathbf{C}$, $C_0 > 0$ such that

$$|p(\rho) - z_0| \geq m(\rho)/C_0, \ \ \rho \in \mathbf{R}^{2n}. \tag{1.4}$$

Let Σ denote the closure of $p(\mathbf{R}^{2n})$ so that $\Sigma = p(\mathbf{R}^{2n}) \cup \Sigma_\infty$, where $\Sigma_\infty \subset \mathbf{C}$ is the set of accumulation points of p in the limit $(x, \xi) = \infty$.

For $h > 0$ small enough, we also let P denote the h-Weyl quantization,

$$Pu(x) = P^w(x, hD_x; h)u(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(x-y)\cdot'} P(\frac{x+y}{2}, \eta; h)u(y)dyd\eta.$$

Let $\Omega \Subset \mathbf{C} \setminus \Sigma_\infty$ be open simply connected and not entirely contained in Σ. Then, as we shall see,

1^o $\sigma(P) \cap \Omega$ is discrete for $h > 0$ small enough,

2^o $\forall\, \epsilon > 0$, $\exists\, h(\epsilon) > 0$, such that

$$\sigma(P) \cap \Omega \subset \Sigma + D(0, \epsilon), \ \ 0 < h \leq h(\epsilon).$$

Here $D(0, \epsilon)$ denotes the open disc in $\mathbf{C}$ with center 0 and radius $\epsilon > 0$ and we equip the operator P with the domain $H(m) := (P - z_0)^{-1}(L^2(\mathbf{R}^n))$, where the operator to the right is the pseudodifferential inverse of $P - z_0$ (see [2] and [7]).

If P is selfadjoint (so that p is real-valued) we have Weyl asymptotics:

For every interval $I \subset \Omega$ with $\mathrm{vol}_{\mathbf{R}^{2n}}(p^{-1}(\partial I)) = 0$, the number $N(P, I)$ of eigenvalues of P in I satisfies

$$N(P, I) = \frac{1}{(2\pi h)^n}(\mathrm{vol}\,(p^{-1}(I)) + o(1)), \ \ h \to 0. \tag{1.5}$$

This result has been proved with increasing generality and precision by J. Chazarain, B. Helffer–D. Robert, and V. Ivrii. (We here follow the presentation of [2] where references to original works can be found. The corresponding developement for selfadjoint partial differential operators in the high energy limit has a long and rich history starting with the work of H. Weyl [14].) A very simple and explicit example is given by the harmonic oscillator $P = \frac{1}{2}((hD)^2 + x^2) : L^2(\mathbf{R}) \to L^2(\mathbf{R})$, $P(x, \xi) = p(x, \xi) = \frac{1}{2}(x^2 + \xi^2)$. In this case the eigenvalues are given by $\lambda_k(h) = (k + \frac{1}{2})h$, $k = 0, 1, 2, ...$

In the non-selfadjoint case, Weyl asymptotics does not always hold. If P is a differential operator with analytic coefficients on the real line, then often the

spectrum is determined by action integrals over complex cycles, having nothing to do with volumes of subsets of real phase space. A simple example of this is given by the non-selfadjoint harmonic oscillator,

$$P=\frac{1}{2}((hD)^2+ix^2):L^2(\mathbf{R})\to L^2(\mathbf{R}), \tag{1.6}$$

whose spectrum is equal to $\{e^{i\pi/4}(k+\frac{1}{2})h;\ k\in\mathbf{N}\}$; This is easy to see by the method of complex scaling, or by applying the general multidimensional result of [11]. In this case, we have $\Sigma_\infty=\emptyset$, and Σ is the closed 1st quadrant. Clearly the number of eigenvalues in an open set $\Gamma\Subset\mathbf{C}$ intersecting the 1st quadrant, whose closure avoids the ray given $\arg z=\frac{\pi}{4}$ is equal to zero while the corresponding Weyl coefficient $\mathrm{vol}(p^{-1}(\Gamma))$ is not. (Further results about the non-selfadjoint harmonic oscillator have been obtained by E.B. Davies and L. Boulton, see [1] and further references given there).

However, in this case and for quite a general class of h-pseudodifferential operators in one dimension, it was shown by one of us in [7] that if we replace the operator P by $P+\delta Q_\omega$, where $0<\delta\ll 1$ varies in a suitable parameter range and Q_ω is a random potential of a suitable type then we do have Weyl asymptotic in the interior of Σ with a probability that is close to 1. The book [4] of M. Embree and L.N. Trefethen as well as the paper [13] by L.N. Trefethen and S.J. Chapman contain (in our opinion) numerical examples where one can see the onset of Weyl-asymptotics after adding small random perturbations.

In this work we establish similar results in arbitrary dimension that we shall now describe. Let $0<\widetilde{m},\widehat{m}\le 1$ be square integrable order functions on $\mathbf{R}^{2n}$ such that $\widetilde{m}$ or $\widehat{m}$ is integrable, and let $\widetilde{S}\in S(\widetilde{m})$, $\widehat{S}\in S(\widehat{m})$ be elliptic symbols. We use the same symbols to denote the h-Weyl quantizations. The operators $\widetilde{S}$, $\widehat{S}$ are then Hilbert-Schmidt with

$$\|\widetilde{S}\|_{\mathrm{HS}},\|\widehat{S}\|_{\mathrm{HS}}\backsim h^{-\frac{n}{2}},$$

where $\backsim$ indicates same order of magnitude. Let $\widetilde{e}_1,\widetilde{e}_2,...$, and $\widehat{e}_1,\widehat{e}_2,...$ be orthonormal bases for $L^2(\mathbf{R}^n)$. Our random perturbation will be

$$Q_\omega=\widehat{S}\circ\sum_{j,k}\alpha_{j,k}(\omega)\widehat{e}_j\widetilde{e}_k^*\circ\widetilde{S}, \tag{1.7}$$

where $\alpha_{j,k}$ are independent complex $\mathcal{N}(0,1)$ random variables, and $\widehat{e}_j\widetilde{e}_k^*u=(u|\widetilde{e}_k)\widehat{e}_j$, $u\in L^2$. In the appendix Section 13 we show that up to a change of the set of independent $\mathcal{N}(0,1)$-laws, the representation (8.1) is independent of the choice of bases $\widehat{e}_j$ and $\widetilde{e}_j$.

Let

$$M = C_1 h^{-n}, \tag{1.8}$$

for some $C_1 \gg 1$. Then, as we shall see in Section 8, we have the following estimate on the probability that Q be large in Hilbert-Schmidt norm:

$$P(\|Q\|_{\mathrm{HS}}^2 \geq M^2) \leq C\exp(-h^{-2n}/C), \tag{1.9}$$

for some new constant $C > 0$. In the following discussion we may restrict the attention to the case when $\|Q_\omega\|_{\mathrm{HS}} \leq M$. We wish to study the eigenvalue distribution of $P + \delta Q_\omega$ for δ in a suitable range.

Let $\Gamma \Subset \Omega$ be open with C^2 boundary and assume that for every $z \in \partial\Gamma$:

$$\begin{gathered}\Sigma_z := p^{-1}(z) \text{ is a smooth sub-manifold of } T^*\mathbf{R}^n \text{ on} \\ \text{which } dp, d\overline{p} \text{ are linearly independent at every point.}\end{gathered} \tag{1.10}$$

The following result will be established in Section 10.

Theorem 1.1 *Let $\Gamma \Subset \Omega$ be open with C^2 boundary and make the assumption (1.10). For $0 < h \ll 1$, let $\delta > 0$ be a small parameter such that*

$$0 < \delta \ll h^{3n+1/2}.$$

For some small parameter $0 < \epsilon \ll 1$ assume $h \ln \delta^{-1} \ll \epsilon \ll 1$ (or equivalently $\delta \geq e^{-\epsilon/(Dh)}$ for some $D \gg 1$, implying also that $\epsilon \geq \widetilde{C} h \ln h^{-1}$ for some $\widetilde{C} > 0$). Then there is a constant $C > 0$ (that is independent of h, δ and ϵ) such that the number $N(P_\delta, \Gamma)$ of eigenvalues of P_δ in Γ satisfies

$$|N(P_\delta, \Gamma) - \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma))| \leq C\frac{\sqrt{\epsilon}}{h^n} \tag{1.11}$$

with probability

$$\geq 1 - \frac{C}{\sqrt{\epsilon}} e^{-\frac{\epsilon/2}{(2\pi h)^n}}.$$

This is a restatement of Theorem 10.1. In Theorem 10.3 we give a similar statement about the simultaneous Weyl asymptotics for all Γs in a family of sets that satisfy the assumptions of the above theorem uniformly. The lower bound on the probability becomes slightly worse but is still very close to 1 for suitable values of ϵ.

The condition (1.10) says that $\partial\Gamma$ does not intersect the set of critical values of p and this is clearly not a serious restriction when $\overline{\Gamma}$ is contained in the interior of Σ.

However, we also would like to study the eigenvalue distribution near the boundary of Σ, and we then need a weaker assumption.

Let $\Gamma \Subset \Omega$ be open with C^∞ boundary. For z in a neighborhood of $\partial\Gamma$ and $0 < s, t \ll 1$, we put

$$V_z(t) = \mathrm{Vol}\,\{\rho \in \mathbf{R}^{2n};\ |p(\rho) - z|^2 \leq t\}. \tag{1.12}$$

Our weak assumption, replacing (1.10) is

$$\exists \kappa \in]0,1],\ \text{such that}\ V_z(t) = \mathcal{O}(t^\kappa),\ \text{uniformly for}\ z \in \mathrm{neigh}\,(\partial\Gamma),\ 0 \leq t \ll 1. \tag{1.13}$$

Here we have written in an informal way "neigh$\,(\partial\Gamma)$" for some neighbourhood of $\partial\Gamma$. Notice that (1.10) implies (1.13) with $\kappa = 1$.

Generically, we will have $\{p, \{p, \overline{p}\}\} \neq 0$ when $p(\rho) \in \partial\Sigma$ and if we assume that

$$\text{at every point of } p^{-1}(0),\ \text{we have}\ \{p, \overline{p}\} \neq 0 \text{ or } \{p, \{p, \overline{p}\}\} \neq 0, \tag{1.14}$$

then as shown in Example 12.1, we have (1.13) with $\delta_0 = 3/4$. (When p is analytic, we believe that (1.13) will always hold with some $\kappa > 0$ but have not consulted with experts in analytic geometry.) Under this more general assumption, we have

Theorem 1.2 *Assume (1.13) and let δ satisfy*

$$0 < \delta \ll h^{3n+1/2}.$$

Let $N(P + \delta Q_\omega, \Gamma)$ be the number of eigenvalues of $P + \delta Q_\omega$ in Γ. Then for every fixed $K > 0$ and for $0 < r \ll 1$:

$$\begin{aligned} &|N(P + \delta Q_\omega, \Gamma) - \frac{1}{(2\pi h)^n} \iint_{p^{-1}(\Gamma)} dx d\xi| \leq \\ &\frac{C}{h^n}\Big(\frac{\epsilon}{r} + C_K\big(r^K + \ln(\frac{1}{r}) \iint_{p^{-1}(\partial\Gamma + D(0,r)} dx d\xi\big)\Big),\ 0 < r \ll 1, \end{aligned} \tag{1.15}$$

with probability

$$\geq 1 - \frac{C}{r} e^{-\frac{\epsilon}{2}(2\pi h)^{-n}} \tag{1.16}$$

provided that

$$h^\kappa \ln \frac{1}{\delta} \ll \epsilon \ll 1, \tag{1.17}$$

or equivalently,

$$e^{-\frac{\epsilon}{C h^\kappa}} \leq \delta,\ C \gg 1,\ \epsilon \ll 1,$$

implying that $\epsilon \geq \widetilde{C} h^\kappa \ln \frac{1}{h}$, for some $\widetilde{C} > 0$.

This is a restatement of Theorem 12.3 and as explained after that theorem, when $\kappa > 1/2$, the integral in the right hand side of (1.15) is $\mathcal{O}(r^{2\kappa-1})$ and it follows that we have Weyl asymptotics with probability close to 1, if we let r be a suitable power of ϵ. To have the same conclusion when $\kappa \leq 1/2$ we can assume that the integral is $\mathcal{O}(r^{\alpha_0})$ for some $\alpha_0 > 0$.

Again we have a similar theorem about the simultaneous asymptotics for $N(P + \delta Q_\omega, \Gamma)$ when Γ varies in a bounded family of domains satisfying all the assumptions uniformly. See Theorem 12.4.

The proofs follow the same general strategy as those of [7] with some essential differences:

We do not use any non-vanishing assumption on the Poisson bracket $\frac{1}{i}\{p, \overline{p}\}$. Instead we work systematically with the operators P^*P and PP^* and their eigenfunctions in order to set up a Grushin-problem.

As in [7] we reduce ourselves to the study of a random holomorphic function, but in the present work this function appears as the determinant of the full operator (essentially) and we need to make some estimates for determinants of random matrices, and especially to prove that such a determinant is not too small with a probability close to 1. Those estimates were sufficiently elementary to be carried out by hand, but we think that future generalizations and improvements will require a careful study of the existing results on random determinants and possibly the derivation of new results in that direction. See the book [5] of V.I. Girko.

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2 Determinants and Grushin problems

Here we mainly follow [12] and give a more explicit formulation of one of the results there. Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{G}_1$, $\mathcal{G}_2$ be complex Hilbert spaces and let $A_{j,k} : \mathcal{H}_k \to \mathcal{G}_j$ be bounded operators depending in a C^1 fashion on the real parameter $t \in]a, b[$. We also assume that $\dot{A}_{j,k}$ are of trace class and continuously dependent of t in the space of such operators. Here "over-dot" means derivative with respect to t.

Proposition 2.1 *([12]) Assume in addition that* $A = (A_{j,k}) : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{G}_1 \times \mathcal{G}_2$ *has a bounded inverse* $B = (B_{j,k})$*, and that* $A_{2,2}$ *and* $B_{1,1}$ *are invertible. (The invertibility of one of* $A_{2,2}$*,* $B_{1,1}$ *implies that of the other.) Then*

$$\mathrm{tr}\,\dot{A}B = \mathrm{tr}\,\dot{A}_{2,2}A_{2,2}^{-1} - \mathrm{tr}\,B_{1,1}^{-1}\dot{B}_{1,1}. \tag{2.1}$$

Proof: We expand

$$\dot{B}_{j,k} = -\sum_{\nu}\sum_{\mu} B_{j,\nu}\dot{A}_{\nu,\mu}B_{\mu,k},$$

that are of the trace class too. In particular,

$$-\dot{B}_{1,1} = B_{1,1}\dot{A}_{1,1}B_{1,1} + B_{1,1}\dot{A}_{1,2}B_{2,1} + B_{1,2}\dot{A}_{2,1}B_{1,1} + B_{1,2}\dot{A}_{2,2}B_{2,1}.$$

Rewrite the right hand side of (2.1):

$$\begin{aligned}
& \mathrm{tr}\,\dot{A}_{2,2}A_{2,2}^{-1} - \mathrm{tr}\,B_{1,1}^{-1}\dot{B}_{1,1} \\
= \; & \mathrm{tr}\,\dot{A}_{2,2}A_{2,2}^{-1} + \mathrm{tr}\,\dot{A}_{1,1}B_{1,1} + \mathrm{tr}\,\dot{A}_{1,2}B_{2,1} + \mathrm{tr}\,B_{1,1}^{-1}B_{1,2}\dot{A}_{2,1}B_{1,1} + \mathrm{tr}\,B_{1,1}^{-1}B_{1,2}\dot{A}_{2,2}B_{2,1} \\
= \; & \mathrm{tr}\,\dot{A}_{2,2}A_{2,2}^{-1} + \mathrm{tr}\,\dot{A}_{1,1}B_{1,1} + \mathrm{tr}\,\dot{A}_{1,2}B_{2,1} + \mathrm{tr}\,\dot{A}_{2,1}B_{1,2} + \mathrm{tr}\,B_{1,1}^{-1}B_{1,2}\dot{A}_{2,2}B_{2,1} \\
= \; & \mathrm{tr}\,\dot{A}_{2,2}(A_{2,2}^{-1} + B_{2,1}B_{1,1}^{-1}B_{1,2}) + \mathrm{tr}\,\dot{A}_{1,1}B_{1,1} + \mathrm{tr}\,\dot{A}_{1,2}B_{2,1} + \mathrm{tr}\,\dot{A}_{2,1}B_{1,2} \\
= \; & \mathrm{tr}\,\dot{A}B.
\end{aligned}$$

Here we used the cyclicity of the trace and for the last equality the fact that

$$B_{2,2} = A_{2,2}^{-1} + B_{2,1}B_{1,1}^{-1}B_{1,2}. \tag{2.2}$$

To check (2.2) we proceed by equivalences:

$$\begin{aligned}
(2.2) \iff \; & A_{2,2}^{-1} = B_{2,2} - B_{2,1}B_{1,1}^{-1}B_{1,2} \\
\iff \; & 1 = A_{2,2}B_{2,2} - A_{2,2}B_{2,1}B_{1,1}^{-1}B_{1,2} \\
\iff \; & 1 = (1 - A_{2,1}B_{1,2}) + A_{2,1}B_{1,1}B_{1,1}^{-1}B_{1,2} \\
\iff \; & 1 = 1.
\end{aligned}$$

Here and in the following, we often denote the identity operator by 1 when the meaning is clear from the context. $\square$

Consider the case $\mathcal{H}_1 = \mathcal{G}_1 = \mathcal{H}$, $\mathcal{H}_2 = \mathcal{G}_2 = \mathbf{C}^N$,

$$A = \mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix}.$$

Assume also that P, $\mathcal{P}$ are invertible. (In the proposition we can permute the indices 1 and 2 and think of P as $A_{2,2}$.) We look for

$$\widetilde{\mathcal{P}} = \begin{pmatrix} P & \widetilde{R}_- \\ \widetilde{R}_+ & \widetilde{R}_{+-} \end{pmatrix} : \mathcal{H} \times \mathbf{C}^N \to \mathcal{H} \times \mathbf{C}^N,$$

that is also invertible, i.e. we should be able to solve uniquely

$$\begin{cases} Pu + \widetilde{R}_- \widetilde{u}_- = v, \\ \widetilde{R}_+ u + \widetilde{R}_{+-} \widetilde{u}_- = \widetilde{v}_+. \end{cases} \tag{2.3}$$

Let

$$\begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix}^{-1}.$$

Rewrite the first equation in (2.3) as $Pu = v - \widetilde{R}_- \widetilde{u}_-$. The general solution to that equation is

$$u = E(v - \widetilde{R}_- \widetilde{u}_-) + E_+ v_+,$$

where v_+, $\widetilde{u}_-$ should be determined so that

$$0 = E_-(v - \widetilde{R}_- \widetilde{u}_-) + E_{-+} v_+. \tag{2.4}$$

The second equation in (2.3) becomes

$$\widetilde{v}_+ = \widetilde{R}_+ E(v - \widetilde{R}_- \widetilde{u}_-) + \widetilde{R}_+ E_+ v_+ + \widetilde{R}_{+-} \widetilde{u}_-. \tag{2.5}$$

Hence we get the following system that is equivalent to (2.3):

$$\begin{cases} E_{-+} v_+ - E_- \widetilde{R}_- \widetilde{u}_- = -E_- v, \\ \widetilde{R}_+ E_+ v_+ + (\widetilde{R}_{+-} - \widetilde{R}_+ E \widetilde{R}_-) \widetilde{u}_- = \widetilde{v}_+ - \widetilde{R}_+ E v, \end{cases} \tag{2.6}$$

so (2.3) is well-posed iff

$$\begin{pmatrix} E_{-+} & -E_- \widetilde{R}_- \\ \widetilde{R}_+ E_+ & \widetilde{R}_{+-} - \widetilde{R}_+ E \widetilde{R}_- \end{pmatrix} : \mathbf{C}^{2N} \to \mathbf{C}^{2N} \tag{2.7}$$

is invertible.

Choose $\widetilde{R}_+ = tR_+$, $\widetilde{R}_- = sR_-$, $\widetilde{R}_{+-} = r\mathrm{id}_{\mathbf{C}^N}$ with $s, t, r \in \mathbf{C}$, and use that $R_+ E_+ = 1$, $E_- R_- = 1$, $R_+ E = 0$, to see that the matrix (2.7) is equal to

$$\begin{pmatrix} E_{-+} & -s \\ t & r \end{pmatrix}. \tag{2.8}$$

This matrix is invertible precisely when (s,t,r) belongs to the set

$$\{(s,t,0);\, st\neq 0\}\cup\{(s,t,r);\, r\neq 0,\, -\frac{st}{r}\notin\sigma(E_{-+})\}. \tag{2.9}$$

Since P is invertible, we know that $0\notin\sigma(E_{-+})$. We can therefore find a C^1-curve

$$[0,1]\ni\tau\mapsto(s(\tau),t(\tau),r(\tau))\in \text{ the set (2.9)},$$

with

$$(s(0),t(0),r(0))=(1,1,0),\quad (s(1),t(1),r(1))=(0,0,1).$$

This means that we have a C^1 deformation

$$\mathcal{P}(\tau)=\begin{pmatrix} P & s(\tau)R_- \\ t(\tau)R_+ & r(\tau)1\end{pmatrix}:\ \mathcal{H}\times\mathbf{C}^N\to\mathcal{H}\times\mathbf{C}^N$$

of bijective operators with

$$\mathcal{P}(0)=\mathcal{P},\ \mathcal{P}(1)=\begin{pmatrix} P & 0\\ 0 & 1\end{pmatrix}.$$

Applying (2.1) with the indices "1" and "2" permuted gives

$$\operatorname{tr}\dot{\mathcal{P}}\mathcal{P}^{-1}=\operatorname{tr}\dot{P}P^{-1}-\operatorname{tr}E_{-+}^{-1}(\tau)\dot{E}_{-+}(\tau)=-\operatorname{tr}E_{-+}^{-1}(\tau)\dot{E}_{-+}(\tau),$$

where now "over-dot" means derivative with respect to τ. If we integrate from $\tau=0$ to $\tau=1$, we get with a suitable choice of branches for ln:

$$\ln\det P-\ln\det\mathcal{P}=\ln\det E_{-+}(0).$$

For this relation to make sense we also assume that

$$P-1 \text{ is of trace class.} \tag{2.10}$$

Then for the original operator and its inverse we have

$$\ln\det P=\ln\det\mathcal{P}+\ln\det E_{-+}, \tag{2.11}$$

or equivalently,

$$\det P=\det\mathcal{P}\det E_{-+}. \tag{2.12}$$

3 General frame-work and reduction to trace class operators

Let $m \geq 1$ be an order function on $\mathbf{R}^{2n}$ in the sense that there exist constants $C_0 \geq 1$, $N_0 > 0$, such that

$$m(\rho) \leq C_0 \langle \rho - \mu \rangle^{N_0} m(\mu),\ \forall \rho, \mu \in \mathbf{R}^{2n},$$

where we write $\langle \rho \rangle = \sqrt{1 + |\rho|^2}$. We consider a symbol

$$P(\rho; h) \sim p(\rho) + hp_1(\rho) + \ldots \text{ in } S(\mathbf{R}^{2n}, m),$$

where

$$S(\mathbf{R}^{2n}, m) = \{a \in C^\infty(\mathbf{R}^{2n});\ |\partial_{x,\xi}^\alpha a(x,\xi)| \leq C_\alpha m(x,\xi),\ \forall (x,\xi) \in \mathbf{R}^{2n},\ \alpha \in \mathbf{N}^{2n}\}.$$

Put

$$\Sigma = \overline{p(\mathbf{R}^{2n})},\ \Sigma_\infty = \{z \in \mathbf{C};\ \exists \rho_j \in \mathbf{R}^{2n},\ j = 1, 2, 3, \ldots,\ \rho_j \to \infty,\ p(\rho_j) \to z,\ j \to \infty\}.$$

Assume $\exists z_0 \in \mathbf{C} \setminus \Sigma$, $C_0 > 0$, such that

$$|p(\rho) - z_0| \geq m(\rho)/C_0,\ \forall \rho \in \mathbf{R}^{2n}. \tag{3.1}$$

Then as pointed out in [7], for every $z \in \mathbf{C} \setminus \Sigma$, there exists $C > 0$ such that

$$|p(\rho) - z| \geq m(\rho)/C,\ \forall \rho \in \mathbf{R}^{2n}, \tag{3.2}$$

and for every $z \in \mathbf{C} \setminus \Sigma_\infty$, there exists $C > 0$ such that

$$|p(\rho) - z| \geq m(\rho)/C,\ \forall \rho \in \mathbf{R}^{2n} \text{ with } |\rho| \geq C. \tag{3.3}$$

Let $\Omega \Subset \mathbf{C} \backslash \Sigma_\infty$ be open simply connected containing at least one point $z_0 \in \mathbf{C} \backslash \Sigma$.

Lemma 3.1 *For every compact set $K \subset \Omega$, there exists a smooth map $\kappa : \Omega \backslash \{z_0\} \to \Omega \backslash \{z_0\}$ with $\kappa(z) = z$ for all z in a neighborhood of $\partial\Omega$, such that $\kappa(\Sigma \cap \Omega) \cap K = \emptyset$.*

Proof: Ω is diffeomorphic to the open unit disc $D(0,1)$ in such a way that z_0 corresponds to 0. Now consider $\widetilde{\kappa} : D(0,1) \setminus \{0\} \to D(0,1) \setminus \{0\}$ defined by $\widetilde{\kappa}(z) = f(|z|)z/|z|$, where f is a smooth function on $]0,1]$ with $1 - \epsilon \leq f(r) \leq 1$, such that $f(r) = r$ for $1 - r \leq \epsilon/2$. Choosing $\epsilon > 0$ small enough and conjugating with the diffeomorphism above we get the desired map κ. $\square$

Let $\widetilde{\Omega} \Subset \Omega$ be open. Take κ as in the lemma with K containing the closure of $\widetilde{\Omega}$. Extend κ to be the identity in $\mathbf{C} \setminus \Omega$ and put $\widetilde{p} = \kappa \circ p$. Then $\widetilde{p}(\rho) - z$ is elliptic in the sense of (3.2), uniformly for $z \in \widetilde{\Omega}$ and

$$\widetilde{p} - p \in C_0^\infty(\mathbf{R}^{2n}). \tag{3.4}$$

Put

$$\widetilde{P} = \widetilde{p} + hp_1 + h^2 p_2 + \dots .$$

Now pass to operators and denote by the same letters symbols and their h-Weyl quantizations. We shall consider P as a closed operator: $L^2 \to L^2$ with domain $H(m) := (P - z_0)^{-1} L^2$ (see [7]). From the discussion above, we get

- For every compact set $K \subset \mathbf{C} \setminus \Sigma$, we have $\sigma(P) \cap K = \emptyset$, when $h > 0$ is small enough.
- $\sigma(P) \cap \widetilde{\Omega}$ is discrete when $h > 0$ is small enough.
- $\sigma(\widetilde{P}) \cap \widetilde{\Omega} = \emptyset$ when $h > 0$ is small enough.

In view of the last property and (3.4), we also have (for $h > 0$ small enough),

Proposition 3.2 *For $z \in \widetilde{\Omega}$, we have that*

$$P(z) := (\widetilde{P} - z)^{-1}(P - z) = 1 + K(z),$$

where $K(z)$ is a trace class operator. Moreover,

$$z \in \sigma(P) \Leftrightarrow 0 \in \sigma(P(z)).$$

Notice that $K(z) = (\widetilde{P} - z)^{-1}(P - \widetilde{P})$ is the quantization of a symbol belonging to the intersection of $S(\widetilde{m})$ for all order functions $\widetilde{m}$.

4 Some functional calculus

Let $P = 1 + K$, $K \in \mathrm{Op}_h(S(m))$, where $m \in C^\infty(\mathbf{R}^{2n};]0, \infty[)$ is an integrable order function. We also assume that $K = k_0 + hk_1 + \dots$ in $S(m)$ on the symbol level. We shall review some functional calculus for $Q = P^* P$ and more generally for a selfadjoint operator $Q \geq 0$ with $Q \sim q + hq_1 + \dots$ on the symbol level, with $Q - 1 \in S(m)$, $q \geq 0$.

Let $\psi \in C_0^\infty(\mathbf{R})$. For $h \ll \alpha \ll 1$ we shall study the properties of $\psi(\alpha^{-1} Q)$.

To this end we shall consider $\alpha^{-1}Q$ as a symbol with h/α as a new semiclassical parameter, after a suitable dilation in phase space. More precisely, we make the change of variables

$$x = \alpha^{\frac{1}{2}}\widetilde{x},\ D_x = \alpha^{-\frac{1}{2}}D_{\widetilde{x}}$$

and write

$$\frac{1}{\alpha}Q(x, hD_x; h) = \frac{1}{\alpha}Q(\alpha^{\frac{1}{2}}(\widetilde{x}, \frac{h}{\alpha}D_{\widetilde{x}}); h), \tag{4.1}$$

with symbol $\alpha^{-1}Q(\alpha^{1/2}(\widetilde{x}, \widetilde{\xi}); h)$ for the h/α-quantization. The lower order terms are $\mathcal{O}(h/\alpha)$ uniformly with all their derivatives, so we shall just look at the leading symbol

$$\frac{q(\alpha^{\frac{1}{2}}(x,\xi))}{\alpha}, \tag{4.2}$$

where we dropped the tildes on the new variables. The (new) associated order function will be

$$m(x,\xi) := 1 + \frac{q(\alpha^{\frac{1}{2}}(x,\xi))}{\alpha} \geq 1. \tag{4.3}$$

We have

$$\nabla m = \frac{(\nabla q)(\alpha^{\frac{1}{2}}(x,\xi))}{\alpha^{\frac{1}{2}}} \leq C\frac{q^{\frac{1}{2}}(\alpha^{\frac{1}{2}}(x,\xi))}{\alpha^{\frac{1}{2}}} \leq Cm(x,\xi)^{\frac{1}{2}},$$
$$\nabla^2 m = \mathcal{O}(1),$$

so by Taylor's formula,

$$m(\rho) = m(\mu) + \mathcal{O}(1)m(\mu)^{\frac{1}{2}}|\rho-\mu| + \mathcal{O}(1)|\rho-\mu|^2,$$

and since $m(\mu) \geq 1$:

$$m(\rho) \leq C\langle\rho-\mu\rangle^2 m(\mu). \tag{4.4}$$

Hence m is an order function, uniformly with respect to α.

Similarly, we have the improved symbol estimates

$$\nabla\frac{q(\alpha^{\frac{1}{2}}\rho)}{\alpha} = \mathcal{O}(1)m^{\frac{1}{2}}, \tag{4.5}$$

$$\nabla^2\frac{q(\alpha^{\frac{1}{2}}\rho)}{\alpha} = \mathcal{O}(1), \tag{4.6}$$

$$\nabla^k\frac{q(\alpha^{\frac{1}{2}}\rho)}{\alpha} = \mathcal{O}(\alpha^{\frac{k}{2}-1}),\ k \geq 2. \tag{4.7}$$

In particular, we have the standard symbol estimates

$$\nabla^k \frac{q(\alpha^{\frac{1}{2}}\rho)}{\alpha} = \mathcal{O}(1)m(\rho). \tag{4.8}$$

It is therefore clear that we can apply the functional calculus in the version of [9] (see also [2]), to see that if $\psi \in C_0^\infty(\mathbf{R})$, and if we interpret Q/α as the right hand side of (4.1), then

$$\psi(\alpha^{-1}Q) = \mathrm{Op}_{\frac{h}{\alpha},\widetilde{x}}(\widetilde{f}), \tag{4.9}$$

where

$$\widetilde{f} = \sum_0^\infty (\frac{h}{\alpha})^\nu f_\nu(\widetilde{x},\widetilde{\xi}),\ \text{in } S(m^{-1}), \tag{4.10}$$

with $f_0(\widetilde{x},\widetilde{\xi}) = \psi(\alpha^{-1}q(\alpha^{1/2}(\widetilde{x},\widetilde{\xi})))$,

$$f_\nu = \sum_{j\le j(\nu)} a_{j,\nu}(\widetilde{x},\widetilde{\xi},\alpha)\psi^{(j)}(\alpha^{-1}q(\alpha^{1/2}(\widetilde{x},\widetilde{\xi}))),\ a_{j,\nu}\in S(1). \tag{4.11}$$

Proposition 4.1 *Let $\widetilde{m} = \widetilde{m}_\alpha(\widetilde{x},\widetilde{\xi})$ be an order function, uniformly with respect to α, such that $\widetilde{m}(\widetilde{x},\widetilde{\xi}) = 1$ for $\alpha^{-1}q(\alpha^{1/2}(\widetilde{x},\widetilde{\xi})) \le \sup\operatorname{supp}\psi + 1/C$, for some $C > 0$ that is independent of α. Then (4.10) holds in $S(\widetilde{m})$, for h and h/α sufficiently small.*

Proof: Write $q_\alpha = q(\alpha^{1/2}(\widetilde{x},\widetilde{\xi}))/\alpha$, $Q_\alpha = \alpha^{-1}Q(\alpha^{1/2}(\widetilde{x},\frac{h}{\alpha}D_{\widetilde{x}});h)$ and drop the tildes. Let $\widehat{q}_\alpha \in S(m)$ be such that $\sup\operatorname{supp}\psi + 1/(5C) \le \widehat{q}_\alpha$, and be equal to q_α when $q_\alpha \ge \sup\operatorname{supp}\psi + 2/(5C)$. Let $\chi_\alpha \in S(1)$ be equal to 1 when $q_\alpha \le \sup\operatorname{supp}\psi + 3/(5C)$ and equal to 0 when $q_\alpha \ge \sup\operatorname{supp}\psi + 4/(5C)$. We use the same symbols q_α, $\widehat{q}_\alpha$, χ_α to denote the h/α quantizations.

Let $\widetilde{\psi}$ be an almost holomorphic extension of ψ and recall the Cauchy-Green-Riemann-Stokes formula, in the operator sense ([9], [2], [3]):

$$\psi(q_\alpha) = \frac{1}{\pi}\int (z-q_\alpha)^{-1}\frac{\partial\widetilde{\psi}(z)}{\partial\overline{z}}L(dz),$$

where $L(dz)$ denotes Lebesgue-measure. For z in a neighborhood of $\operatorname{supp}\widetilde{\psi}$, we write

$$(z-q_\alpha)^{-1} = (z-q_\alpha)^{-1}\circ\chi_\alpha + (z-\widehat{q}_\alpha)^{-1}\circ(1-\chi_\alpha) - (z-q_\alpha)^{-1}(\widehat{q}_\alpha - q_\alpha)(z-\widehat{q}_\alpha)^{-1}(1-\chi_\alpha).$$

Then, since the middle term is holomorphic near the support of $\widetilde{\psi}$,

$$\psi(q_\alpha) = \psi(q_\alpha) \circ \chi_\alpha - \frac{1}{\pi} \int (z - q_\alpha)^{-1} (\widehat{q}_\alpha - q_\alpha)(z - \widehat{q}_\alpha)^{-1} (1 - \chi_\alpha) \frac{\partial \widetilde{\psi}}{\partial \overline{z}} L(dz).$$

Here the symbol of $\psi(q_\alpha) \circ \chi_\alpha$ has the asymptotic expansion (4.10) in $S(\widetilde{m})$, thanks to the extra factor χ_α and with the same terms given by (4.11). For $z \in \mathrm{neigh}\,(\mathrm{supp}\,\widetilde{\psi})$, $(z - \widehat{q}_\alpha)^{-1} \in \mathrm{Op}\,(1/m)$ depends holomorphically on z and thanks to the factor $\widehat{q}_\alpha - q_\alpha$, whose support on the symbol level is separated from that of $1 - \chi_\alpha$ by some fixed positive distance, we know that $(\widehat{q}_\alpha - q_\alpha)(z - \widehat{q}_\alpha)^{-1}(1 - \chi_\alpha) \in (h/\alpha)^N \mathrm{Op}\,(S(\widetilde{m}))$ for any $N \geq 0$ and any $\widetilde{m}$ as in the proposition. Combining this with the estimates for the symbol of $(z - q_\alpha)^{-1}$ from the Beals lemma as in [9] (see also [2], Proposition 8.6), we get the proposition. $\square$

We next apply the functional result to the study of certain determinants. Let $\chi \in C_0^\infty([0, +\infty[; [0, +\infty[)$, $\chi(0) > 0$ and extend χ to $C_0^\infty(\mathbf{R}; \mathbf{C})$ in such a way that $\chi(t) > 0$ near 0 and $t + \chi(t) \neq 0$, $\forall t \in \mathbf{R}$. We want to study $\ln \det(Q + \alpha \chi(\alpha^{-1} Q))$, when $h \ll \alpha \ll 1$. Let us first recall from [10] that if $\widetilde{Q} = \mathrm{Op}_h(\widetilde{q})$ with $\widetilde{q} \in S(1)$, $\widetilde{q} > 0$, $\widetilde{q} - 1 \in S(\widetilde{m})$, where $\widetilde{m}$ is an integrable order function, then

$$\ln \det \widetilde{Q} = \frac{1}{(2\pi h)^n} (\iint \ln \widetilde{q}(x, \xi) dx d\xi + \mathcal{O}(h)). \tag{4.12}$$

In fact, let $\widetilde{Q}_t = (1 - t)1 + t\widetilde{Q}$, so that $\widetilde{Q}_0 = 1$, $\widetilde{Q}_1 = \widetilde{Q}$. Then by standard elliptic calculus, with $\widetilde{q}_t = (1 - t) + t\widetilde{q}$, we have

$$\begin{aligned}
\frac{d}{dt} \ln \det \widetilde{Q}_t &= \operatorname{tr} \widetilde{Q}_t^{-1} \frac{d}{dt} \widetilde{Q}_t \\
&= \frac{1}{(2\pi h)^n} (\iint \widetilde{q}_t^{-1} \frac{d}{dt} \widetilde{q}_t \, dx d\xi + \mathcal{O}(h)) \\
&= \frac{1}{(2\pi h)^n} (\frac{d}{dt} \iint \ln \widetilde{q}_t(x, \xi) dx d\xi + \mathcal{O}(h)),
\end{aligned}$$

and integrating from $t = 0$ to $t = 1$, we get (4.12).

For $\alpha = \alpha_1 > 0$ fixed with $\alpha_1 \ll 1$, this applies to $Q + \alpha_1 \chi(\alpha_1^{-1} Q)$ and we get

$$\ln \det(Q + \alpha_1 \chi(\alpha_1^{-1} Q)) = \frac{1}{(2\pi h)^n} (\iint \ln(q + \alpha_1 \chi(\alpha_1^{-1} q)) dx d\xi + \mathcal{O}(h)). \tag{4.13}$$

We have for $t > 0$ and $E \geq 0$:

$$\frac{d}{dt} \ln(E + t\chi(\frac{E}{t})) = \frac{1}{t} \psi(\frac{E}{t}),$$

with
$$\psi(E)=\frac{\chi(E)-E\chi'(E)}{E+\chi(E)}.$$
Now for $h\ll\alpha\le t\le\alpha_1$, we get from Proposition 4.1 by dilatation:
$$\begin{aligned}&\frac{d}{dt}\ln\det(Q+t\chi(t^{-1}Q))=\operatorname{tr}t^{-1}\psi(t^{-1}Q) \qquad (4.14)\\ &=\Big(\frac{t}{2\pi h}\Big)^n\Big(\iint\frac{1}{t}\psi(\frac{q(t^{\frac12}(\widetilde{x},\widetilde{\xi}))}{t})d\widetilde{x}d\widetilde{\xi}+\mathcal{O}(\frac{h}{t})\frac{1}{t}\iint\widehat{\chi}(\frac{q(t^{\frac12}(\widetilde{x},\widetilde{\xi}))}{t})d\widetilde{x}d\widetilde{\xi}\\ &\quad+\mathcal{O}(1)\frac{1}{t}(\frac{h}{t})^\infty\iint(1+\operatorname{dist}((\widetilde{x},\widetilde{\xi});\operatorname{supp}\widehat{\chi}(\frac{q(t^{\frac12}(\cdot))}{t})))^{-N}d\widetilde{x}d\widetilde{\xi}\Big),\end{aligned}$$
where $0\le\widehat{\chi}\in C_0^\infty(\mathbf{R})$ is equal to one on some interval containing $[0,\sup\operatorname{supp}\psi]$ and the last term is coming from the "remainder" in the asymptotic development (4.10).

We are interested in the integral of this quantity from $t=\alpha$ to $t=\alpha_1$. Let us first treat the leading term
$$\begin{aligned}(\frac{t}{2\pi h})^n\iint\frac{1}{t}\psi(\frac{q(t^{\frac12}(\widetilde{x},\widetilde{\xi}))}{t})d\widetilde{x}d\widetilde{\xi}&=\frac{1}{(2\pi h)^n}\iint\frac{1}{t}\psi(\frac{q(x,\xi)}{t})dxd\xi\\ &=\frac{1}{(2\pi h)^n}\iint\frac{d}{dt}\ln(q+t\chi(\frac{q}{t}))dxd\xi,\end{aligned}$$
and integrating this from $t=\alpha$ to $t=\alpha_1$, we get
$$\Big[\frac{1}{(2\pi h)^n}\iint\ln(q+t\chi(\frac{q}{t}))dxd\xi\Big]_{t=\alpha}^{\alpha_1}. \qquad (4.15)$$
The second term in (4.14) is
$$\begin{aligned}\mathcal{O}(1)(\frac{t}{h})^n\frac{h}{t^2}\iint\widehat{\chi}\left(\frac{1}{t}q\big(t^{\frac12}(\widetilde{x},\widetilde{\xi})\big)\right)d\widetilde{x}d\widetilde{\xi}&=\mathcal{O}(1)\frac{1}{h^n}\frac{h}{t^2}\iint\widehat{\chi}\big(\frac{1}{t}q(x,\xi)\big)dxd\xi\\ &\le\mathcal{O}(1)h^{-n}\frac{h}{t^2}\iint_{q(x,\xi)\le C^2t}dxd\xi.\end{aligned}$$
Integrating this from $t=\alpha$ to $t=\alpha_1$, we get
$$\begin{aligned}&\mathcal{O}(1)h^{1-n}\iint\int_{\max(\alpha,q/C)\le t\le\alpha_1}\frac{1}{t^2}dtdxd\xi \qquad (4.16)\\ &=\mathcal{O}(1)h^{1-n}\iint_{q(x,\xi)\le C\alpha_1}\left(\frac{1}{\max\big(\alpha,q(x,\xi)/C\big)}-\frac{1}{\alpha_1}\right)dxd\xi\\ &\le\mathcal{O}(1)h^{-n}\iint_{q(x,\xi)\le C\alpha_1}\frac{h}{\alpha+q(x,\xi)}dxd\xi\qquad.\end{aligned}$$

When estimating the third term in (4.14) we consider separately the regions $|(x,\xi)| \leq C$ and $|(x,\xi)| > C$ for some large $C \geq 1$. Consider first the region $|(x,\xi)| \leq C$. Put

$$d_t(\widetilde{x},\widetilde{\xi}) = \mathrm{dist}\,((\widetilde{x},\widetilde{\xi}), \{(y,\eta);\ \frac{1}{t} q(t^{\frac{1}{2}}(y,\eta)) \leq \widehat{C}\}),\ \ \widehat{C} = \sup \mathrm{supp}\, \widehat{\chi}.$$

For (y,η) with $q(t^{\frac{1}{2}}(y,\eta))/t \leq \widehat{C}$, we have $\nabla(q(t^{\frac{1}{2}}(y,\eta))/t) = \mathcal{O}(1)$ and $\nabla^2(q(t^{\frac{1}{2}}(y,\eta))/t) = \mathcal{O}(1)$, so by Taylor expanding at (y,η) we get

$$\frac{1}{t} q(t^{\frac{1}{2}}(\widetilde{x},\widetilde{\xi})) \leq \mathcal{O}(1)(1 + d_t(\widetilde{x},\widetilde{\xi}) + d_t(\widetilde{x},\widetilde{\xi})^2) \leq \mathcal{O}(1)(1 + d_t(\widetilde{x},\widetilde{\xi}))^2.$$

The contribution to the third term in (4.14) from $t^{1/2}|(\widetilde{x},\widetilde{\xi})| \leq C$ is therefore

$$\begin{aligned} &\mathcal{O}_N(1)(\frac{h}{t})^{\infty} \frac{1}{t} \iint_{|(\widetilde{x},\widetilde{\xi})| \leq Ct^{-1/2}} (1 + \frac{1}{t} q(t^{\frac{1}{2}}(\widetilde{x},\widetilde{\xi})))^{-N} d\widetilde{x} d\widetilde{\xi} \\ &= \mathcal{O}_{M,N}(1) \frac{1}{h^n} (\frac{h}{t})^M \frac{1}{t} \iint_{|(x,\xi)| \leq C} (1 + \frac{1}{t} q(x,\xi))^{-N} dx d\xi. \end{aligned} \tag{4.17}$$

We integrate this from $t = \alpha$ to α_1, so we want to estimate

$$h^M \int_{\alpha}^{\alpha_1} \frac{1}{t^{M+1}(1 + \frac{1}{t} q(x,\xi))^N} dt.$$

If $q \leq \alpha$, we get

$$\mathcal{O}(1) h^M \int_{\alpha}^{\alpha_1} \frac{1}{t^{M+1}} dt = \mathcal{O}(1)((\frac{h}{\alpha})^M).$$

If $\alpha < q \leq \alpha_1$, we get

$$h^M \int_{\alpha}^{q} \frac{1}{t^{M+1}} (1 + \frac{q}{t})^{-N} dt + h^M \int_{q}^{\alpha_1} \frac{1}{t^{M+1}} (1 + \frac{q}{t})^{-N} dt \backsim h^M \int_{\alpha}^{q} \frac{t^N}{t^{M+1} q^N} dt + (\frac{h}{q})^M,$$

with the symbol $\backsim$ indicating same order of magnitude. Choose $N = M + 1$, to get

$$\backsim (\frac{h}{q})^M.$$

For $\alpha_1 \leq q$, we get

$$\mathcal{O}(1) h^M \int_{\alpha}^{\alpha_1} \frac{t^N}{t^{M+1} q^N} dt = \mathcal{O}(1) h^M$$

with the same choice of N. Thus the expression (4.17) is

$$\frac{\mathcal{O}(1)}{h^n}\iint_{|(x,\xi)|\le C}\Big(\frac{h}{\alpha+q(x,\xi)}\Big)^M dxd\xi,\ \forall M\ge 0.$$

We next look at the contribution to the last term in (4.14) from the region $t^{\frac{1}{2}}|(\widetilde{x},\widetilde{\xi})|>C$, which is

$$\begin{aligned}\mathcal{O}(1)\Big(\frac{h}{t}\Big)^{\infty}\frac{1}{t}\iint_{t^{\frac{1}{2}}|(\widetilde{x},\widetilde{\xi})|\ge C}(1+|(\widetilde{x},\widetilde{\xi})|)^{-N}d\widetilde{x}d\widetilde{\xi}&=\mathcal{O}(1)(\frac{h}{t})^M t^{\widetilde{N}},\ \forall M,\widetilde{N},\\ &=\mathcal{O}(1)h^M,\ \forall M.\end{aligned}$$

Summing up, we have proved

Proposition 4.2 *Let $\chi\in C_0^\infty(\mathbf{R})$ with $\chi(0)>0$ and $\chi(t)\ge 0$ for $t\ge 0$. Then for $h\ll\alpha\ll 1$:*

$$\begin{aligned}&\ln\det(Q+\alpha\chi(\alpha^{-1}Q))= \qquad (4.18)\\ &\frac{1}{(2\pi h)^n}\Big(\iint\ln(q+\alpha\chi(\frac{q}{\alpha}))dxd\xi+\mathcal{O}(1)\iint_{|(x,\xi)|\le C}\frac{h}{\alpha+q(x,\xi)}dxd\xi\Big)+\mathcal{O}(h^\infty).\end{aligned}$$

Most of the proof was based on (4.14), of which the second part is valid for any $\psi\in C_0^\infty(\mathbf{R})$. The estimates leading to the preceding proposition, also give

Proposition 4.3 *Let $\chi\in C_0^\infty(\mathbf{R})$, and choose $\widehat{\chi}\in C_0^\infty(\mathbf{R};[0,1])$ equal to 1 on an interval containing 0 and* $\operatorname{supp}\chi$*. Then for $0<h\ll\alpha\ll 1$, we have*

$$\begin{aligned}\operatorname{tr}\chi(\alpha^{-1}Q) &= \frac{1}{(2\pi h)^n}\Big(\iint\chi(\frac{q(x,\xi)}{\alpha})dxd\xi+\mathcal{O}(\frac{h}{\alpha})\iint\widehat{\chi}(\frac{q(x,\xi)}{\alpha})dxd\xi \qquad (4.19)\\ &\quad+\mathcal{O}_{N,M}(1)(\frac{h}{\alpha})^M\iint_{|(x,\xi)|\le C}(1+\frac{q}{\alpha})^{-N}dxd\xi+\mathcal{O}(h^\infty)\Big).\end{aligned}$$

Put

$$V(t)=\iint_{q(x,\xi)\le t}dxd\xi,\ 0\le t\le\frac{1}{2}, \qquad (4.20)$$

so that $0\le V(t)$ is an increasing function. By introducing an assumption on $V(t)$, we shall make (4.19) more explicit and replace (4.18) by a more explicit estimate.

We assume that there exists $\kappa\in]0,1]$ such that

$$V(t)=\mathcal{O}(1)t^\kappa,\ 0\le t\le\frac{1}{2}. \qquad (4.21)$$

The first integral in (4.18) can be written

$$\begin{aligned}\iint \ln(q+\alpha\chi(\frac{q}{\alpha}))dxd\xi &= \iint \ln(q)dxd\xi + \mathcal{O}(1)\iint_{q\le C\alpha} \ln\frac{1}{q}dxd\xi \\ &= \iint \ln(q)dxd\xi + \mathcal{O}(1)\int_0^{C\alpha} \ln\frac{1}{q}dV(q),\end{aligned}$$

so (4.18) gives

$$\begin{aligned}&\ln\det(Q+\alpha\chi(\frac{1}{\alpha}Q)) \qquad (4.22)\\ =\ & \frac{1}{(2\pi h)^n}(\iint \ln(q)dxd\xi + \mathcal{O}(1)\int_0^{C\alpha}\ln\frac{1}{q}dV(q) + \mathcal{O}(1)\int_0^{\frac{1}{2}}\frac{h}{\alpha+q}dV(q) + \mathcal{O}(h^\infty)).\end{aligned}$$

Similarly (4.19) can be written

$$\begin{aligned}\operatorname{tr}\chi(\alpha^{-1}Q) &= \frac{1}{(2\pi h)^n}\Big(\int \chi(\frac{q}{\alpha})dV(q) + \mathcal{O}(\frac{h}{\alpha})\int \widehat{\chi}(\frac{q}{\alpha})dV(q) \\ &\quad + \mathcal{O}_{N,M}(1)(\frac{h}{\alpha})^M\int_0^{\alpha_1}(1+\frac{q}{\alpha})^{-N}dV(q) + \mathcal{O}(h)\Big).\end{aligned} \tag{4.23}$$

In particular, the number $N(\alpha)$ of eigenvalues of Q in $[0,\alpha]$ satisfies

$$N(\alpha) \le \mathcal{O}(1)\big(h^{-n}\int_0^{\alpha_1}(1+\frac{q}{\alpha})^{-N}dV(q) + h^{1-n}). \tag{4.24}$$

Proposition 4.4 *Under the assumption (4.21), we have*

$$\int_0^\alpha \ln(q)dV(q) = \mathcal{O}(\alpha^\kappa \ln\alpha), \tag{4.25}$$

$$\int_0^{\alpha_1}\frac{h}{\alpha+q}dV(q) = \begin{cases}\mathcal{O}(\alpha^\kappa\frac{h}{\alpha}), \ \textit{for } \kappa<1, \\ \\ \mathcal{O}(h\ln\frac{1}{\alpha}), \ \textit{when } \kappa=1,\end{cases} \tag{4.26}$$

$$N(\alpha) = \mathcal{O}(\alpha^\kappa h^{-n} + h^{1-n}). \tag{4.27}$$

Proof: This follows by straight forward calculations, starting with an integration by parts.

$$\int_0^\alpha \ln(q)dV(q) = [\ln(q)V(q)]_0^\alpha - \int_0^\alpha \frac{1}{q}V(q)dq = \mathcal{O}(\alpha^\kappa \ln(\alpha)),$$

$$\begin{aligned}
\int_0^{\alpha_1} \frac{h}{\alpha+q}dV(q) &= [\frac{h}{\alpha+q}V(q)]_0^{\alpha_1} + \int_0^{\alpha_1} \frac{h}{(\alpha+q)^2}V(q)dq \\
&= \mathcal{O}(h) + \mathcal{O}(1)\int_0^{\alpha_1} \frac{hq^\kappa}{(\alpha+q)^2}dq \\
&= \mathcal{O}(h) + \mathcal{O}(1)h\alpha^{\kappa-1}\int_0^{\alpha_1/\alpha} \frac{\widetilde{q}^\kappa}{(1+\widetilde{q})^2}d\widetilde{q} \\
&= \begin{cases} \mathcal{O}(h\alpha^{\kappa-1}),\ 0<\kappa<1, \\ \\ \mathcal{O}(h\ln\frac{1}{\alpha}),\ \kappa=1. \end{cases}
\end{aligned}$$

To get (4.27), we use (4.24) and the estimate

$$\begin{aligned}
\int_0^{\alpha_1} (1+\frac{q}{\alpha})^{-N}dV(q) &= [(1+\frac{q}{\alpha})^{-N}V(q)]_0^{\alpha_1} + N\int_0^{\alpha_1}(1+\frac{q}{\alpha})^{-(N+1)}V(q)\frac{dq}{\alpha} \\
&= \mathcal{O}(1)\alpha^N + \mathcal{O}(1)\int_0^\infty (1+\widetilde{q})^{-(N+1)}\widetilde{q}^\kappa d\widetilde{q}\alpha^\kappa \\
&= \mathcal{O}(1)\alpha^\kappa.
\end{aligned}$$

$\square$

Since we will always assume that $h \ll \alpha \ll 1$, and since $\kappa \le 1$, we can simplify (4.27) to

$$N(\alpha) = \mathcal{O}(1)\alpha^\kappa h^{-n}. \tag{4.28}$$

Proposition 4.5 *Assume (4.21). Under the assumptions of Proposition 4.2, we have*

$$\ln\det(Q+\alpha\chi(\alpha^{-1}Q)) = \frac{1}{(2\pi h)^n}(\iint \ln(q)dxd\xi + \mathcal{O}(1)\alpha^\kappa\ln\alpha). \tag{4.29}$$

Under the assumptions of Proposition 4.3, we have

$$\operatorname{tr}\chi(\alpha^{-1}Q) = \frac{1}{(2\pi h)^n}(\iint \chi(\frac{q(x,\xi)}{\alpha})dxd\xi + \mathcal{O}(1)\alpha^\kappa\frac{h}{\alpha}). \tag{4.30}$$

For $0 < h \ll \alpha \ll 1$, the number $N(\alpha)$ of eigenvalues of Q in $[0,\alpha]$ satisfies (4.27).

Notice that when $Q \geq 0$:

$$\ln \det Q \leq \ln \det(Q + \alpha\chi(\alpha^{-1}Q)),$$

so (4.29) with $\alpha = Ch$, $C \gg 1$, gives an upper bound which is more precise than the one in [10].

5 Grushin problem for the unperturbed operator

In Section 3 we introduced the operator

$$P(z) = 1 + K(z), \quad K(z) \in \mathrm{Op}_h(S(m))$$

where m is an integrable order function, so that $K(z)$ is a trace class operator. Here $P(z)$ depends holomorphically on $z \in \widetilde{\Omega} \Subset \mathbf{C}$, where $\widetilde{\Omega}$ is open. Also recall that $P(z) = (\widetilde{P} - z)^{-1}(P - z)$. We are interested in the spectrum of small random perturbations of P; $P_\delta = P + \delta Q$. Correspondingly, we get $P_\delta(z) = (\widetilde{P} - z)^{-1}(P_\delta - z) = 1 + K_\delta(z)$, and the main work in later sections will be to study $|\det(1 + K_\delta(z))|$. The upper bounds will be fairly simple to get, and the delicate point will be to get lower bounds. As a preparation for this more delicate step, we here study a Grushin problem for the unperturbed operator $P(z)$. In this section $z \in \widetilde{\Omega}$ will be fixed and we simply write P instead $P(z)$.

Let $e_1, e_2, ...$ be an orthonormal (ON) basis of eigenvectors of P^*P and let $0 \leq \lambda_1 \leq \lambda_2 \leq ...$ be the corresponding eigenvalues. (Strictly speaking, if we want the eigenvalues to form an increasing sequence, the set of indices j should be of the form $J = J_0 \cup J_1 \cup J_2$, with

- $J_0 = \mathbf{N}$ or a finite set, $0 \leq \lambda_j < 1$, for $j \in J_0$,
- $J_1 = \mathbf{N}$ or a finite set, $\lambda_j = 1$, $j \in J_1$
- $J_2 = -\mathbf{N}$ or a finite set, $\lambda_j > 1$, $j \in J_2$.

We will only be concerned with finitely many indices from J_0.)

Since $P = P(z)$ is a Fredholm operator of index zero by Proposition 3.2, we know that PP^* and P^*P have the same number N_0 of eigenvalues equal to 0. Let $f_1, ..., f_{N_0}$ be an ON basis of $\ker(PP^*)$. For $j > N_0$, we have $\lambda_j > 0$ and Pe_j is an eigenvector of PP^* with eigenvalue λ_j:

$$PP^*Pe_j = \lambda_j Pe_j.$$

Using standard notations for norms and scalar products, we put $f_j = \|Pe_j\|^{-1} Pe_j$. Then $\{f_j\}_{j\in J}$ is an ON system of eigenvectors of PP^*, with $PP^* f_j = \lambda_j f_j$. Let $f \in L^2(\mathbf{R}^n)$ with $(f|f_j) = 0$ for all $j \in J$. Then $(P^* f|e_j) = (f|Pe_j)$. If $j \le N_0$, we get $(P^* f|e_j) = (f|0) = 0$, and if $j \ge N_0 + 1$, we get $(P^* f|e_j) = \|Pe_j\|(f|f_j) = 0$. Hence $P^* f = 0$, so $f \in \ker(PP^*)$ and hence $f = 0$ is zero since $\ker(PP^*)$ is the span of $f_1, ..., f_{N_0}$. We conclude that $\{f_j\}_{j\in J}$ is an ON *basis* of eigenvectors of PP^*. By construction, we have $Pe_j = w_j f_j$ with $0 \le w_j = \|Pe_j\|$. Then

$$w_j^2 = (P^* Pe_j|e_j) = \lambda_j,$$

so $w_j = \sqrt{\lambda_j}$ and it follows that,

$$Pe_j = \sqrt{\lambda_j} f_j, \tag{5.1}$$

$$P^* f_j = \sqrt{\lambda_j} e_j \tag{5.2}$$

for all $j \in J$.

Let $0 < \alpha \ll 1$ and let $N = N(\alpha)$ be given by

$$\lambda_j \le \alpha \Leftrightarrow j \le N(\alpha). \tag{5.3}$$

Define

$$R_+ : L^2(\mathbf{R}^n) \to \mathbf{C}^N, \quad R_- : \mathbf{C}^N \to L^2(\mathbf{R}^n)$$

by

$$R_+ u = \sqrt{\alpha}((u|e_j))_{j=1}^N, \quad R_- u_- = \sqrt{\alpha} \sum_1^N u_-(j) f_j,$$

and put

$$\mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} : L^2(\mathbf{R}^n) \times \mathbf{C}^N \to L^2(\mathbf{R}^n) \times \mathbf{C}^N. \tag{5.4}$$

If $u = \sum_{j\in J} u_j e_j$, $u_- = (u_-(j))_{j=1}^N$, we get

$$\mathcal{P} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} \sum_J \sqrt{\lambda_j} u_j f_j + \sum_1^N \sqrt{\alpha} u_-(j) f_j \\ (\sqrt{\alpha} u_j)_{j=1}^N \end{pmatrix},$$

and we conclude that

$$\begin{aligned} |\det \mathcal{P}| &= (\prod_1^N |\det \begin{pmatrix} \sqrt{\lambda_j} & \sqrt{\alpha} \\ \sqrt{\alpha} & 0 \end{pmatrix} |)(\prod_{N<j\in J} \sqrt{\lambda_j}) \\ &= \alpha^N \prod_{N<j\in J} \sqrt{\lambda_j} \\ &= \alpha^{\frac{N}{2}} \prod_J \max(\sqrt{\alpha}, \sqrt{\lambda_j}). \end{aligned} \tag{5.5}$$

Notice that

$$|\det P| = \prod_J \sqrt{\lambda_j}. \tag{5.6}$$

Let $\delta_j(k) = \delta_{j,k},\ 1 \le j,k \le N$. Then $\mathcal{P}$ maps $\mathbf{C}e_j \times \mathbf{C}\delta_j$ to $\mathbf{C}f_j \times \mathbf{C}\delta_j$ and has the corresponding matrix

$$\begin{pmatrix} \sqrt{\lambda_j} & \sqrt{\alpha} \\ \sqrt{\alpha} & 0 \end{pmatrix}.$$

The inverse is given by

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{\alpha}} \\ \frac{1}{\sqrt{\alpha}} & -\frac{\sqrt{\lambda_j}}{\alpha} \end{pmatrix},$$

so if $v = \sum_J v_j f_j$, $v_+ = \sum_1^N v_+(j)\delta_j$, then (writing as before $\mathcal{E}$ for $\mathcal{P}^{-1}$)

$$\mathcal{E}\begin{pmatrix} v \\ v_+ \end{pmatrix} = \begin{pmatrix} \sum_{N+1}^{\infty} \frac{1}{\sqrt{\lambda_j}} v_j e_j + \frac{1}{\sqrt{\alpha}} \sum_1^N v_+(j) e_j \\ \sum_1^N \frac{1}{\sqrt{\alpha}} v_j \delta_j - \sum_1^N \frac{\sqrt{\lambda_j}}{\alpha} v_+(j)\delta_j \end{pmatrix} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \begin{pmatrix} v \\ v_+ \end{pmatrix}, \tag{5.7}$$

where

$$E_+ v_+ = \frac{1}{\sqrt{\alpha}} \sum_1^N v_+(j) e_j, \tag{5.8}$$

$$E_- v = \frac{1}{\sqrt{\alpha}} \sum_1^N v_j \delta_j,$$

$$E_{-+} = -\frac{1}{\alpha} \operatorname{diag}(\sqrt{\lambda_j}),$$

$$\|E\|, \|E_+\|, \|E_-\|, \|E_{-+}\| \le \frac{1}{\sqrt{\alpha}}.$$

From (5.5)–(5.8), we see that

$$|\det P| = |\det \mathcal{P}||\det E_{-+}|, \tag{5.9}$$

as we already know from (2.12).

We next study $|\det \mathcal{P}|$, when $h \ll \alpha \ll 1$. The formula (5.5) can be written

$$|\det \mathcal{P}|^2 = \alpha^N \det 1_\alpha(P^* P), \tag{5.10}$$

where $1_\alpha(t) = \max(\alpha, t)$. Let $\chi \in C_0^\infty([0,2[;[0,1])$ be equal to 1 on $[0,1]$. Then for $t \ge 0$,

$$t + \frac{\alpha}{4}\chi(\frac{4t}{\alpha}) \le 1_\alpha(t) \le t + \alpha\chi(\frac{t}{\alpha}). \tag{5.11}$$

In the following, we assume that $Q = P^*P$ satisfies the assumptions of Section 4, including (4.21), and choose $h \ll \alpha \ll 1$. Then we know that

$$N(\alpha) = \mathcal{O}(\alpha^\kappa h^{-n}), \tag{5.12}$$

and Proposition 4.5 in combination with (5.10)–(5.12) show that

$$\ln|\det \mathcal{P}|^2 = \frac{1}{(2\pi h)^n}(\iint \ln(q)dxd\xi + \mathcal{O}(1)\alpha^\kappa \ln \alpha). \tag{5.13}$$

As noticed after Proposition 4.5, we also have the upper bound

$$\ln\det P^*P \le \frac{1}{(2\pi h)^n}(\iint \ln(q)dxd\xi + \mathcal{O}(1)\alpha^\kappa \ln\frac{1}{\alpha}). \tag{5.14}$$

6 The Hilbert-Schmidt norm of a Gaussian random matrix.

Let $\alpha(\omega)$ be a complex Gaussian random variable with density

$$\frac{1}{\pi\sigma^2}e^{-|\alpha|^2/\sigma^2}L(d\alpha),\ \ L(d\alpha) = d\mathrm{Re}\,\alpha\, d\,\mathrm{Im}\,\alpha, \tag{6.1}$$

that is a $\mathcal{N}(0,\sigma^2)$-law with σ^2 denoting the variance. The distribution of $|\alpha(\omega)|^2$ is

$$\mu d\alpha = \frac{1}{s}e^{-r/s}H(r)dr, \tag{6.2}$$

where $s = \sigma^2$ and $H(r)$ denotes the standard Heaviside function. Notice that

$$\||\alpha|^2\|_{L^1} = \langle|\alpha|^2\rangle = \sigma^2.$$

Let $\alpha_j(\omega)$, $j = 1, 2, ...$ be independent random variables as above with variance σ_j^2 and assume for simplicity that $\sigma_1 \ge \sigma_j$ for all j. We also assume that

$$\sum_1^\infty \sigma_j^2 < \infty, \tag{6.3}$$

implying the a.s. convergence of $\sum_1^\infty |\alpha_j(\omega)|^2$.

We want to estimate the probability that $\sum |\alpha_j(\omega)|^2 \geq a$. The probability distribution of $\sum_1^\infty |\alpha_j(\omega)|^2$ is equal to $(\mu_1 * \mu_2 * ....)dx$, where μ_j is given in (6.2) with $s = s_j = \sigma_j^2$, so that

$$\sum_1^\infty s_j < \infty. \tag{6.4}$$

The Fourier transform of μ_j is given by

$$\widehat{\mu}_j(\rho) = \frac{1}{1 + is_j\rho}, \tag{6.5}$$

which has a simple pole at $\rho = i/s_j$. The probability that we are after, is

$$\int_a^\infty (\mu_1 * \mu_2 * ...)dr = \frac{1}{2\pi} \int \prod_1^\infty (\widehat{\mu}_j(\rho)) \overline{\widehat{1_{[a,\infty[}}}(\rho) d\rho, \tag{6.6}$$

by Parseval's identity. Here

$$\widehat{1_{[a,\infty[}}(\rho) = \frac{1}{i(\rho - i0)} e^{-ia\rho}, \tag{6.7}$$

so the probability (6.5) becomes

$$\frac{i}{2\pi} \int_{-\infty}^\infty (\prod_1^\infty \frac{1}{1 + is_j\rho}) \frac{1}{\rho + i0} e^{ia\rho} d\rho. \tag{6.8}$$

The assumption (6.4) implies that the infinite product converges away from the poles i/s_j. For ρ in a half plane $\operatorname{Im}\rho \leq b < \frac{1}{2s_1}$, we have

$$|\frac{1}{1 + is_1\rho}| \leq \frac{1}{((1 - bs_1)^2 + s_1^2(\operatorname{Re}\rho)^2)^{\frac{1}{2}}},$$

$$|\prod_2^\infty \frac{1}{1 + is_j\rho}| \leq \prod_2^\infty \frac{1}{1 - bs_j} \leq \exp{(C_0 \sum_2^\infty bs_j)},$$

where C_0 is a universal constant appearing in the estimate,

$$\frac{1}{1 - t} \leq e^{C_0 t},\ 0 \leq t \leq \frac{1}{2}.$$

Shifting the contour in (6.8) from $\mathbf{R}$ to $\mathbf{R}+ib$ and choosing $b=1/(2s_1)$, we can estimate the probability (6.6) from above by

$$C(s_1)\mathrm{exp}\,[\frac{C_0}{2s_1}\sum_1^\infty s_j-\frac{1}{2s_1}a], \tag{6.9}$$

where $C_0>0$ is the universal constant introduced above and $C(s_1)$ can be chosen uniformly bounded on any compact subset of $]0,+\infty]$.

Remark 6.1 W. Bordeaux Montrieux has used a more elementary argument in the case of real matrices, by means of the Markov-Chebyschev inequality. With P denoting the probability and $\langle\rangle$ the expectation values, it gives for every $a>0$:

$$a\mathsf{P}(\sum_1^\infty|\alpha_j(\omega)|^2\ge a)\le\langle\sum_1^\infty|\alpha_j(\omega)|^2\rangle=\sum_1^\infty\langle|\alpha_j(\omega)|^2\rangle=\sum_1^\infty\sigma_j^2. \tag{6.10}$$

We will prefer (6.9) however, since it gives an exponential decay with respect to a.

Remark 6.2 If $Q=(\alpha_{j,k}(\omega))_{j,k\in\mathbf{N}}$ is a random matrix where $\alpha_{j,k}(\omega)$ are independent $\mathcal{N}(0,\sigma_{j,k}^2)$ laws, and

$$\sum_{j,k}\sigma_{j,k}^2<\infty, \tag{6.11}$$

then (6.9) gives an estimate on the probability that the Hilbert-Schmidt norm is $\ge a^{1/2}$:

$$\mathsf{P}(\|(\alpha_{j,k}(\omega))\|_{\mathrm{HS}}^2\ge a)\le C(s_1)\mathrm{exp}\,[\frac{C_0}{2s_1}\sum_{j,k\in\mathbf{N}^2}\sigma_{j,k}^2-\frac{1}{2s_1}a] \tag{6.12}$$

where C_0, $C(s_1)$ is the same constants as in (6.9) and $s_1=\max\sigma_{j,k}^2$.

7 Estimates on determinants of Gaussian random matrices

Consider first a random vector

$${}^t u(\omega)=(\alpha_1(\omega),...,\alpha_N(\omega))\in\mathbf{C}^N, \tag{7.1}$$

where $\alpha_1,...,\alpha_N$ are independent complex Gaussian random variables with a $\mathcal{N}(0,1)$ law and ω is the random parameter living in a probability space with probability P. The law of α_j, i.e. the direct image of P under α_j, is given by

$$(\alpha_j)_*(\mathsf{P})=\frac{1}{\pi}e^{-|z|^2}L(dz)=:f(z)L(dz) \tag{7.2}$$

and $L(dz) = L_{\mathbf{C}}(dz)$ is the Lebesgue measure on $\mathbf{C}$.

The distribution of u is

$$u_*(\mathsf{P}) = \frac{1}{\pi^N} e^{-|u|^2} L_{\mathbf{C}^N}(du). \tag{7.3}$$

If $U : \mathbf{C}^N \to \mathbf{C}^N$ is unitary, then Uu has the same distribution as u.

We next compute the distribution of $|u(\omega)|^2$. The distribution of $|\alpha_j(\omega)|^2$ is $\mu(r)dr$, where

$$\mu(r) = -H(r)\frac{d}{dr}e^{-r} = e^{-r}H(r),$$

where $H(r) = 1_{[0,\infty[}(r)$. We have $\widehat{\mu}(\rho) = \frac{1}{1+i\rho}$.

We have $|u(\omega)|^2 = \sum_1^N |\alpha_j(\omega)|^2$ and since $|\alpha_j(\omega)|^2$ are independent and identically distributed, the distribution of $|u(\omega)|^2$ is $\mu * ... * \mu \, dr = \mu^{*N} dr$, where $*$ indicates convolution. For $r > 0$, we get by straight forward calculation the χ^2_{2N} distribution (for the variable $2r$)

$$\mu^{*N} dr = \frac{r^{N-1}e^{-r}}{(N-1)!} H(r) dr. \tag{7.4}$$

Recall here that

$$\int_0^\infty r^{N-1} e^{-r} dr = \Gamma(N) = (N-1)!,$$

so μ^{*N} is indeed normalized.

The expectation value of each $|\alpha_j(\omega)|^2$ is 1 so:

$$\langle |u(\omega)|^2 \rangle = N. \tag{7.5}$$

We next estimate the probability that $|u(\omega)|^2$ is very large in a fashion that is slightly different from that of Section 6. It will be convenient to pass to the variable $\ln(|u(\omega)|^2)$, which has the distribution obtained from (7.4) by replacing r by $t = \ln r$, so that $r = e^t$, $dr/r = dt$. Thus $\ln(|u(\omega)|^2)$ has the distribution

$$\frac{r^N e^{-r}}{(N-1)!} H(r) \frac{dr}{r} = \frac{e^{Nt - e^t}}{(N-1)!} dt =: \nu_N(t) dt. \tag{7.6}$$

Now consider a random matrix

$$(u_1 ... u_N) \tag{7.7}$$

where $u_k(\omega)$ are random vectors in $\mathbf{C}^N$ (here viewed as column vectors) of the form

$${}^t u_k(\omega) = (\alpha_{1,k}(\omega), ..., \alpha_{N,k}(\omega)),$$

and all the $\alpha_{j,k}$ are independent with the same law (7.2).

Then

$$\det(u_1\, u_2 ... u_N) = \det(u_1\, \widetilde{u}_2 ... \widetilde{u}_N), \tag{7.8}$$

where $\widetilde{u}_j$ are obtained in the following way (assuming the u_j to be linearly independent, as they are almost surely): $\widetilde{u}_2$ is the orthogonal projection of u_2 in the orthogonal complement $(u_1)^\perp$, $\widetilde{u}_3$ is the orthogonal projection of u_3 in $(u_1, u_2)^\perp = (u_1, \widetilde{u}_2)^\perp$, etc.

If u_1 is fixed, then $\widetilde{u}_2$ can be viewed as a random vector in $\mathbf{C}^{N-1}$ of the type (7.1), (7.2), and with u_1, u_2 fixed, we can view $\widetilde{u}_3$ as a random vector of the same type in $\mathbf{C}^{N-2}$ etc. On the other hand

$$|\det(u_1\, u_2 ... u_N)|^2 = |u_1|^2 |\widetilde{u}_2|^2 \cdot .. \cdot |\widetilde{u}_N|^2. \tag{7.9}$$

The squared lengths $|u_1|^2, |\widetilde{u}_2|^2, ..., |\widetilde{u}_N|^2$ are independent random variables with distributions $\mu^{*N} dr, \mu^{*(N-1)} dr, ..., \mu dr$. This reduction plays an important role in [5]. The following lemma will not be used directly.

Lemma 7.1 *Let $\alpha, \beta > 0$ be independent random variables with distributions $\mu_\alpha(r)\frac{dr}{r}$, $\mu_\beta(r)\frac{dr}{r}$. Then the product $\alpha\beta$ has the distribution $\mu_{\alpha\beta}\frac{dr}{r}$, with*

$$\mu_{\alpha\beta} = \mu_\alpha \sharp \mu_\beta := \mathcal{M}^{-1}((\mathcal{M}\mu_\alpha)(\mathcal{M}\mu_\beta)). \tag{7.10}$$

Here

$$\mathcal{M}\mu(\tau) = \int r^{-i\tau} \mu(r) \frac{dr}{r}$$

is the Mellin transform of μ.

Proof: Recall that the Mellin transform of $\mu(r)$ is the Fourier transform of $\mu(e^t)$; $r = e^t$, $r^{-1} dr = dt$. The distribution of $\ln \alpha$ is related to that of α by the same change of variables $\mu_\alpha(r)\frac{dr}{r} \to \mu_\alpha(e^t) dt = \nu_\alpha(t) dt$. Since multiplication on the Fourier transform side corresponds to convolution, (7.10) is equivalent to the fact that the distribution of the sum of two independent random variables is equal to the convolution of the distributions of the two variables. $\Box$

The proof also shows that the multiplicative convolution in the lemma is given by

$$\mu_\alpha \sharp \mu_\beta(r) = \int_0^\infty \mu_\alpha(\frac{r}{\rho}) \mu_\beta(\rho) \frac{d\rho}{\rho}. \tag{7.11}$$

As already mentioned we shall not use the lemma directly but rather its proof by taking logarithms and use that the distribution of the random variable $\ln |\det(u_1\, u_2 ... u_N)|^2$ is equal to

$$(\nu_1 * \nu_2 * ... * \nu_N) dt, \tag{7.12}$$

with ν_j defined in (7.6).

We have

$$\nu_N(t) \leq \widetilde{\nu}_N(t) := \frac{1}{(N-1)!} e^{Nt}.$$

Choose $x(N) \in \mathbf{R}$ such that

$$\int_{-\infty}^{x(N)} \widetilde{\nu}_N(t) dt = 1. \tag{7.13}$$

More explicitely, we have

$$\frac{1}{N!} e^{Nx(N)} = 1, \quad x(N) = \frac{1}{N} \ln(N!) = \frac{1}{N} \ln \Gamma(N+1). \tag{7.14}$$

Using Stirling's formula,

$$\frac{(N-1)!}{\sqrt{2\pi}} = \frac{\Gamma(N)}{\sqrt{2\pi}} = e^{-N} N^{N-\frac{1}{2}} (1 + \mathcal{O}(\frac{1}{N})),$$

we get

$$\begin{aligned} x(N) &= \frac{1}{N}\big(\frac{1}{2}\ln(2\pi) - (N+1) + (N+\frac{1}{2})\ln(N+1) + \mathcal{O}(\frac{1}{N})\big) \qquad (7.15) \\ &= \frac{1}{N}\big((N+\frac{1}{2})\ln N - N + C_0 + \mathcal{O}(\frac{1}{N})\big) \\ &= \ln N + \frac{1}{2N}\ln N - 1 + \frac{C_0}{N} + \mathcal{O}(\frac{1}{N^2}), \end{aligned}$$

where $C_0 = (\ln 2\pi)/2 > 0$.

With this choice of $x(N)$, we put

$$\rho_N(t) = 1_{]-\infty, x(N)]}(t) \widetilde{\nu}_N(t),$$

so that $\rho_N(t)dt$ is a probability measure "obtained from $\nu_N(t)dt$, by transfering mass to the left" in the sense that

$$\int f \nu_N dt \leq \int f \rho_N dt, \tag{7.16}$$

whenever f is a bounded decreasing function. Equivalently,

$$g * \nu_N \leq g * \rho_N,$$

whenever g is a bounded increasing function. Now, for such a g, both $g * \nu_N$ and $g * \rho_N$ are bounded increasing functions, so by induction, we get

$$g * \nu_1 * ... * \nu_N \le g * \rho_1 * ... * \rho_N.$$

In particular, by taking $g = H$, we get

$$\int_{-\infty}^{t} (\nu_1 * ... * \nu_N)(s) ds \le \int_{-\infty}^{t} (\rho_1 * ... * \rho_N)(s) ds, \ t \in \mathbf{R}. \tag{7.17}$$

We have by (7.14)

$$\begin{aligned} \widehat{\rho}_N(\tau) &= \int_{-\infty}^{x(N)} \frac{1}{(N-1)!} e^{t(N-i\tau)} dt = \frac{1}{(N-1)!(N-i\tau)} e^{Nx(N)-ix(N)\tau} \quad (7.18) \\ &= \frac{e^{-ix(N)\tau}}{1-i\frac{\tau}{N}}. \end{aligned}$$

This function has a pole at $\tau = -iN$.

Similarly,

$$\widehat{1_{]-\infty,a]}}(\tau) = \frac{i}{\tau + i0} e^{-ia\tau}. \tag{7.19}$$

By Parseval's formula, we get

$$\begin{aligned} \int_{-\infty}^{a} \rho_1 * .. * \rho_N dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\rho_1 * .. * \rho_N)(\tau) \overline{\mathcal{F} 1_{]-\infty,a]}}(\tau) dt \quad (7.20) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\tau(\sum_1^N x(j)-a)} \frac{-i}{\tau - i0} \prod_1^N \frac{1}{(1-\frac{i\tau}{j})} d\tau. \quad (7.21) \end{aligned}$$

We deform the contour to $\operatorname{Im} \tau = -1/2$ (half-way between $\mathbf{R}$ and the first pole in the lower half-plane). For $j \ge 2$, we use the estimate

$$|\frac{1}{1-\frac{i\tau}{j}}| \le \frac{1}{1-\frac{1}{2j}} = \exp\left(\frac{1}{2j} + \mathcal{O}(\frac{1}{j^2})\right),$$

when $\operatorname{Im} \tau = -1/2$. Hence,

$$\prod_2^N |\frac{1}{1-\frac{i\tau}{j}}| \le \exp\left(\frac{1}{2} \sum_2^N (\frac{1}{j} + \frac{\mathcal{O}(1)}{j^2})\right) \le C N^{\frac{1}{2}}.$$

It follows that for $a \leq \sum_1^N x(j)$:

$$\int_{-\infty}^{a} \rho_1 * .. * \rho_N dt \leq C N^{\frac{1}{2}} \exp\left(-\frac{1}{2}(\sum_1^N x(j) - a)\right). \tag{7.22}$$

In view of (7.17), (7.20) the right hand side is an upper bound for the probability that $\ln |\det(u_1...u_N)|^2 \leq a$.

From the formula (7.15), we get for some constants $C_1, C_2 \in \mathbf{R}$:

$$\sum_1^N x(j) \geq C_1 + (N + \frac{1}{2}) \ln N - 2N + \frac{1}{4}(\ln N)^2 + C_0 \ln N \geq C_2 + (N + \frac{1}{2}) \ln N - 2N. \tag{7.23}$$

Hence, for $a \leq C_2 + (N + \frac{1}{2}) \ln N - 2N$,

$$\begin{aligned} & \mathsf{P}(\ln |\det(u_1...u_N)|^2 \leq a) \\ \leq & \; C N^{\frac{1}{2}} \exp[-\frac{1}{2}(C_2 + (N + \frac{1}{2}) \ln N - 2N - a)] \\ = & \; C \exp[-\frac{1}{2}(C_2 + (N - \frac{1}{2}) \ln N - 2N - a)]. \end{aligned} \tag{7.24}$$

We shall next extend our bounds on the probability for the determinant to be small, to determinants of the form

$$\det(D + Q)$$

where $Q = (u_1...u_N)$ is as before, and $D = (d_1...d_N)$ is a fixed complex $N \times N$ matrix. As before, we can write

$$|\det((d_1 + u_1)...(d_N + u_N))|^2 = |d_1 + u_1|^2 |\widetilde{d}_2 + \widetilde{u}_2|^2 \cdot .. \cdot |\widetilde{d}_N + \widetilde{u}_N|^2,$$

where $\widetilde{d}_2 = \widetilde{d}_2(u_1)$, $\widetilde{u}_2 = \widetilde{u}_2(u_1, u_2)$ are the orthogonal projections of d_2, u_2 on $(d_1 + u_1)^\perp$, $\widetilde{d}_3 = \widetilde{d}_3(u_1, u_2)$, $\widetilde{u}_3 = \widetilde{u}_3(u_1, u_2, u_3)$ are the orthogonal projections of d_2, u_2 on $(d_1 + u_1, d_2 + u_2)^\perp$ and so on.

Let $\nu_d^{(N)}(t)dt$ be the probability distribution of $\ln |d + u|^2$, when $d \in \mathbf{C}^N$ is fixed and $u \in \mathbf{C}^N$ is random as in (7.1), (7.2). Notice that $\nu_0^{(N)}(t) = \nu^{(N)}(t)$ is the density we have alreay studied.

Lemma 7.2 *For every $a \in \mathbf{R}$, we have*

$$\int_{-\infty}^{a} \nu_d^{(N)}(t) dt \leq \int_{-\infty}^{a} \nu^{(N)}(t) dt.$$

Proof: Equivalently, we have to show that $\mathsf{P}(|d+u|^2 \le \widetilde{a}) \le \mathsf{P}(|u|^2 \le \widetilde{a})$ for every $\widetilde{a} > 0$. For this, we may assume that $d = (c, 0, ..., 0)$, $c > 0$. We then only have to prove that

$$\mathsf{P}(|c + \mathrm{Re}\, u_1|^2 \le b^2) \le \mathsf{P}(|\mathrm{Re}\, u_1|^2 \le b^2),\ b > 0,$$

and here we may replace P by the corresponding probability density

$$\mu(t)dt = \frac{1}{\sqrt{\pi}} e^{-t^2} dt$$

for $\mathrm{Re}\, \mu_1$. Thus, we have to show that

$$\frac{1}{\sqrt{\pi}} \int_{|c+t| \le b} e^{-t^2} dt \le \frac{1}{\sqrt{\pi}} \int_{|t| \le b} e^{-t^2} dt. \tag{7.25}$$

Fix b and rewrite the left hand side as

$$I(c) = \frac{1}{\sqrt{\pi}} \int_{-b-c}^{b-c} e^{-t^2} dt.$$

The derivative satisfies (recall that $c > 0$)

$$I'(c) = \frac{1}{\sqrt{\pi}} (e^{-(b+c)^2} - e^{-(b-c)^2}) \le 0.$$

hence $c \mapsto I(c)$ is decreasing and (7.25) follows, since it is trivially fulfilled when $c = 0$. $\square$

Now consider the probability that $\ln |\det(D+Q)|^2 \le a$. If $\chi_a(t) = H(a-t)$, this probability becomes

$$\int .. \int \mathsf{P}(du_1)...\mathsf{P}(du_N) \times$$
$$\chi_a(\ln |d_1 + u_1|^2 + \ln |\widetilde{d}_2(u_1) + \widetilde{u}_2(u_1, u_2)|^2 + ... + \ln |\widetilde{d}_N(u_1, .., u_{N-1}) + \widetilde{u}_N(u_1, .., u_N)|^2).$$

Here we first carry out the integration with respect to u_N, noticing that with the other $u_1, .., u_{N-1}$ fixed, we may consider $\widetilde{d}_N(u_1, .., u_{N-1})$ as a fixed vector in $\mathbf{C} \simeq (d_1 + u_1, ..., d_{N-1} + u_{N-1})^\perp$ and $\widetilde{u}_N$ as a random vector in $\mathbf{C}$. Using also the lemma, we get

$$\begin{aligned}
& \mathsf{P}(\ln |\det(D+Q)|^2 \le a) \\
= & \int .. \int \nu^{(1)}_{\widetilde{d}_N}(t_N) dt_N \mathsf{P}(du_{N-1})..\mathsf{P}(du_1) \times \\
& \chi_a(\ln |d_1 + u_1|^2 + .. + \ln |\widetilde{d}_{N-1}(u_1, .., u_{N-2}) + \widetilde{u}_{N-1}(u_1, .., u_{N-1})|^2 + t_N) \\
\le & \int .. \int \nu^{(1)}(t_N) dt_N \mathsf{P}(du_{N-1})..\mathsf{P}(du_1) \times \\
& \chi_a(\ln |d_1 + u_1|^2 + .. + \ln |\widetilde{d}_{N-1}(u_1, .., u_{N-2}) + \widetilde{u}_{N-1}(u_1, .., u_{N-1})|^2 + t_N).
\end{aligned}$$

We next estimate the u_{N-1}- integral in the same way and so on. Eventually, we get

Proposition 7.3 *Under the assumptions above,*

$$\begin{aligned}\mathsf{P}(\ln|\det(D+Q)|^2 \leq a) &\leq \int ..\int \chi_a(t_1+...+t_N)\nu^{(1)}(t_N)\nu^{(2)}(t_{N-1})..\nu^{(N)}(t_1)\\ &= \mathsf{P}(\ln|\det Q|^2 \leq a).\end{aligned}$$

In particular the estimate (7.24) extends to random perturbations of constant matrices:

$$\mathsf{P}(\ln|\det(D+Q)|^2 \leq a) \leq C\exp\left[-\frac{1}{2}(C_2+(N-\frac{1}{2})\ln N - 2N - a)\right], \tag{7.26}$$

when $a \leq C_2 + (N+\frac{1}{2})\ln N - 2N$.

8 Grushin problem for the perturbed operator

Let P be as in Section 3. Let $0 < \widetilde{m}, \widehat{m} \leq 1$ be square integrable order functions on $\mathbf{R}^{2n}$ such that $\widetilde{m}$ or $\widehat{m}$ is integrable, and let $\widetilde{S} \in S(\widetilde{m})$, $\widehat{S} \in S(\widehat{m})$ be elliptic symbols. We use the same symbols to denote the h-Weyl quantizations. The operators $\widetilde{S}$, $\widehat{S}$ will be Hilbert-Schmidt with

$$\|\widetilde{S}\|_{\mathrm{HS}}, \|\widehat{S}\|_{\mathrm{HS}} \backsim h^{-\frac{n}{2}}.$$

Let $\widetilde{e}_1, \widetilde{e}_2, ...,$ and $\widehat{e}_1, \widehat{e}_2, ...$ be orthonormal bases for $L^2(\mathbf{R}^n)$. Our random perturbation will be

$$Q_\omega = \widehat{S} \circ \sum_{j,k} \alpha_{j,k}(\omega)\widehat{e}_j\widetilde{e}_k^* \circ \widetilde{S}, \tag{8.1}$$

where $\alpha_{j,k}$ are independent complex $\mathcal{N}(0,1)$ random variables. See the appendix, Section 13 for a general discussion.

Consider the polar decompositions

$$\widehat{S} = \widehat{D}\widehat{U},\ \ \widetilde{S} = \widetilde{U}\widetilde{D}, \tag{8.2}$$

where $\widehat{U}$, $\widetilde{U}$ are unitary pseudodifferential operators with symbol in $S(1)$ and $\widehat{D}$, $\widetilde{D}$ are positive selfadjoint elliptic pseudodifferential operators with symbol in $S(\widehat{m})$ and $S(\widetilde{m})$ respectively. After replacing $\widehat{e}_j$ by $\widehat{U}\widehat{e}_j$ and $\widetilde{e}_k$ by $\widetilde{U}^*\widetilde{e}_k$, we get with the new orthonormal bases that

$$Q_\omega = \widehat{D} \circ \sum_{j,k} \alpha_{j,k}(\omega)\widehat{e}_j\widetilde{e}_k^* \circ \widetilde{D}, \tag{8.3}$$

Now as we recall in the appendix (Section 13), we may replace the bases $\widehat{e}_j$ and $\widetilde{e}_j$ by any new orthonormal bases we like, if we replace the $\alpha_{j,k}(\omega)$ by a new set of random variables (that we also denote by $\alpha_{j,k}$) having identical properties. If we choose $\widehat{e}_j$ to be an orthonormal basis of eigenfunctions of $\widehat{D}$ and similarly for $\widetilde{e}_j$, then we get

$$Q_\omega = \sum_{j,k} \widehat{s}_j \widetilde{s}_k \alpha_{j,k}(\omega) \widehat{e}_j \widetilde{e}_k^*, \tag{8.4}$$

where $\widehat{s}_j > 0$ and $\widetilde{s}_j > 0$ are the eigenvalues of $\widehat{D}$ and $\widetilde{D}$ respectively, i.e. the singular values of $\widehat{S}$ and $\widetilde{S}$.

We are then precisely in the situation of Section 6, noting that $\widehat{s}_j \widetilde{s}_k \alpha_{j,k}(\omega)$ are independent $\mathcal{N}(0, \widehat{s}_j^2 \widetilde{s}_k^2)$-laws, so (6.12) can be applied with $\sigma_{j,k} = \widehat{s}_j \widetilde{s}_k$,

$$\sum_{j,k} \sigma_{j,k}^2 = \|\widehat{S}\|_{\mathrm{HS}}^2 \|\widetilde{S}\|_{\mathrm{HS}}^2 \backsim h^{-2n}.$$

We also know that

$$s_1 = \max \sigma_{j,k} = \|\widehat{S}\| \|\widetilde{S}\| \backsim 1.$$

From (6.12), we deduce that

$$\mathsf{P}(\|Q_\omega\|_{\mathrm{HS}}^2 \geq a) \leq C \exp\,[Ch^{-2n} - \frac{a}{C}] \tag{8.5}$$

for some constant $C > 0$. Let

$$M = C_1 h^{-n}, \tag{8.6}$$

for some $C_1 \gg 1$. Then (8.5) gives

$$\mathsf{P}(\|Q\|_{\mathrm{HS}}^2 \geq M^2) \leq C \exp\,(-h^{-2n}/C), \tag{8.7}$$

for some new constant $C > 0$.

We also want to control the trace class norm of Q_ω, so we will use the assumption that one of $\widetilde{m}$ and $\widehat{m}$ is integrable. Assume for instance that $\widehat{m}$ is integrable. Then $\widehat{m}^{1/2}$ is square integrable and we can factorize $\widehat{S} = \widehat{S}_1 \widehat{S}_2$, with $\widehat{S}_j \in \mathrm{Op}(\widehat{m}^{1/2})$ being Hilbert-Schmidt operators. Let us write

$$Q_\omega = \widehat{S}_1 \widehat{S}_2 \sum_{j,k} \alpha_{j,k}(\omega) \widehat{e}_j \widetilde{e}_k^* \widetilde{S}.$$

Now recall that the composition of two Hilbert-Schmidt operators is of trace class and the corresponding trace class norm does not exceed the product of the Hilbert-Schmidt norms of the two factors. Knowing that $\|\widehat{S}_1\|_{\mathrm{HS}} = \mathcal{O}(h^{-n/2})$ and applying (8.7) to $\widehat{S}_2 \sum_{j,k} \alpha_{j,k}(\omega) \widehat{e}_j \widetilde{e}_k^* \widetilde{S}$, we get

$$\mathsf{P}(\|Q_\omega\|_{\mathrm{tr}} \geq M^{3/2}) \leq C \exp\,(-h^{-2n}/C). \tag{8.8}$$

In the following we will restrict the attention to Q_ω's with

$$\|Q_\omega\|_{\mathrm{HS}} \leq M,\ \|Q_\omega\|_{\mathrm{tr}} \leq M^{3/2}, \tag{8.9}$$

and we have just seen that the probability that this is the case is bounded from below by $1 - Ce^{-h^{-2n}/C}$.

We wish to study the eigenvalue distribution of

$$P_\delta = P - \delta Q_\omega, \tag{8.10}$$

when $\delta > 0$ is sufficiently small. (The minus sign is for notational convenience only.)

Recall from Section 3, that for $z \in \widetilde{\Omega}$,

$$P(z) = (\widetilde{P} - z)^{-1}(P - z) \tag{8.11}$$

is a trace class perturbation of the identity. We now introduce

$$P_\delta(z) = (\widetilde{P} - z)^{-1}(P - \delta Q_\omega - z) = P(z) - \delta(\widetilde{P} - z)^{-1}Q_\omega. \tag{8.12}$$

The Grushin problem will be used to find lower bounds for $|\det P_\delta(z)|$. First we derive an upper bound: We have with $P_\delta(z) = P_\delta$, $P = P(z)$:

$$\begin{aligned}
&P_\delta^* P_\delta && (8.13)\\
&= P^*P - \delta(P^*(\widetilde{P} - z)^{-1}Q_\omega + Q_\omega^*(\widetilde{P}^* - \overline{z})^{-1}P - \delta Q_\omega^*(\widetilde{P}^* - \overline{z})^{-1}(\widetilde{P} - z)^{-1}Q_\omega)\\
&= P^*P + \delta R,
\end{aligned}$$

where

$$\begin{aligned}
\|R\|_{\mathrm{HS}} &\leq C(\|Q_\omega\|_{\mathrm{HS}} + \delta\|Q_\omega\|\|Q_\omega\|_{\mathrm{HS}}) \leq \widetilde{C}M, && (8.14)\\
\|R\|_{\mathrm{tr}} &\leq C(\|Q_\omega\|_{\mathrm{tr}} + \delta\|Q_\omega\|\|Q_\omega\|_{\mathrm{tr}}) \leq \widetilde{C}M^{3/2},
\end{aligned}$$

provided that $\delta\|Q_\omega\| \leq \mathcal{O}(1)$, as will follow from (8.15).

In Section 4 we studied $P^*P + \alpha\chi(\alpha^{-1}P^*P)$ for $h \ll \alpha \ll 1$. This operator is $\geq \alpha$ if $1_{[0,1]} \leq \chi$, as we may assume. Now assume that

$$\delta M \ll h. \tag{8.15}$$

Then

$$P^*P + \alpha\chi(\alpha^{-1}P^*P) + \delta R \geq \frac{\alpha}{2},$$

and

$$\begin{aligned}\ln\det P_\delta^* P_\delta &\le \ln\det(P_\delta^* P_\delta + \alpha\chi(\frac{P^*P}{\alpha}))\\ &= \ln\det(P^*P + \alpha\chi(\frac{P^*P}{\alpha}) + \delta R)\\ &= \ln\det(P^*P + \alpha\chi(\frac{P^*P}{\alpha})) + \int_0^\delta \mathrm{tr}\,((P^*P + \alpha\chi(\frac{P^*P}{\alpha}) + tR)^{-1}R)dt.\end{aligned}$$

The integral is $\mathcal{O}(1)\frac{\delta}{\alpha}\|R\|_{\mathrm{tr}} = \mathcal{O}(1)\delta M^{\frac{3}{2}}/\alpha$ and combining this with (4.29) (assuming now (4.21)), we get

$$\ln\det P_\delta^* P_\delta \le \frac{1}{(2\pi h)^n}(\iint \ln|p|^2 dxd\xi + \mathcal{O}(1)\alpha^\kappa \ln\frac{1}{\alpha}) + \mathcal{O}(1)\frac{\delta M^{\frac{3}{2}}}{\alpha}. \tag{8.16}$$

Here we choose $\alpha = Ch$, $C \gg 1$ and we can drop the last remainder term if we assume that

$$\delta M^{\frac{3}{2}} \ll h^{1+\kappa-n}\ln\frac{1}{h},\ \delta \ll h^{1+\kappa+n/2}\ln\frac{1}{h}. \tag{8.17}$$

For $n \ge 2$ this follows from (8.15), but for $n = 1$ it might be a stronger assumption depending on the value of κ. Then

$$\ln|\det P_\delta| \le \frac{1}{(2\pi h)^n}(\iint \ln|p| dxd\xi + \mathcal{O}(1)h^\kappa \ln\frac{1}{h}). \tag{8.18}$$

Still with $h \ll \alpha \ll 1$ we define R_+, R_-, $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{E}_0 = \mathcal{E}$ as in Section 5. Here z is fixed, $P = P(z)$. With $P_\delta = P_\delta(z)$, we put

$$\mathcal{P}_\delta = \begin{pmatrix} P_\delta & R_- \\ R_+ & 0 \end{pmatrix} :\ L^2(\mathbf{R}^n) \times \mathbf{C}^N \to L^2(\mathbf{R}^n) \times \mathbf{C}^N. \tag{8.19}$$

Now $\|\delta(\widetilde{P} - z)^{-1}Q_\omega\| \le C\delta M$, and

$$\frac{\delta M}{\sqrt{\alpha}} \ll \frac{h}{\sqrt{\alpha}} \ll 1$$

under the assumption (8.15), so $\mathcal{P}_\delta$ has the inverse

$$\mathcal{E}_\delta = \mathcal{E}_0\Big(1 - \begin{pmatrix} \delta(\widetilde{P} - z)^{-1}Q_\omega & 0 \\ 0 & 0 \end{pmatrix} \mathcal{E}_0\Big)^{-1} \tag{8.20}$$

of norm $\le \mathcal{O}(1/\sqrt{\alpha})$. Writing

$$\widetilde{Q}_\omega = (\widetilde{P} - z)^{-1} Q_\omega, \tag{8.21}$$

we have the Neumann series expansion

$$\begin{aligned} \mathcal{E}_\delta &= \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=0}^\infty E^0(\delta \widetilde{Q}_\omega E^0)^j & \sum_{j=0}^\infty (E^0 \delta \widetilde{Q}_\omega)^j E_+^0 \\ \sum_{j=0}^\infty E_-^0(\delta \widetilde{Q}_\omega E^0)^j & E_{-+}^0 + \sum_{j=1}^\infty E_-^0(\delta \widetilde{Q}_\omega E^0)^{j-1} \delta \widetilde{Q}_\omega E_+^0 \end{pmatrix}. \end{aligned} \tag{8.22}$$

For $0 \le t \le \delta$ we have

$$\begin{aligned} \frac{d}{dt} \ln \det \mathcal{P}_t &= -\mathrm{tr}\, \mathcal{E}_t \begin{pmatrix} \widetilde{Q}_\omega & 0 \\ 0 & 0 \end{pmatrix} \\ &= \mathcal{O}(1) \frac{1}{\sqrt{\alpha}} \| \widetilde{Q}_\omega \|_{\mathrm{tr}} \\ &= \mathcal{O}(1) \frac{1}{\sqrt{\alpha}} M^{\frac{3}{2}}, \end{aligned}$$

so

$$\ln \det \mathcal{P}_\delta = \ln \det(\mathcal{P}) + \mathcal{O}(1) \frac{\delta}{\sqrt{\alpha}} M^{\frac{3}{2}}.$$

Applying (5.13), we get

$$\ln|\det \mathcal{P}_\delta| = \frac{1}{(2\pi h)^n} (\iint \ln|p| dx d\xi + \mathcal{O}(1) \alpha^\kappa \ln \alpha) + \mathcal{O}(1) \frac{\delta}{\sqrt{\alpha}} M^{\frac{3}{2}}. \tag{8.23}$$

Again, under the strengthened assumption (8.17), we get with $\alpha = Ch$, $C \gg 1$,

$$\ln|\det \mathcal{P}_\delta| = \frac{1}{(2\pi h)^n} (\iint \ln|p| dx d\xi + \mathcal{O}(1) h^\kappa \ln \frac{1}{h}). \tag{8.24}$$

The idea to get a lower bound for $\ln|\det P_\delta|$ with high probability is now to use (2.11), (2.12) which gives

$$\ln|\det P_\delta| = \ln|\det \mathcal{P}_\delta| + \ln|\det E_{-+}^\delta|, \tag{8.25}$$

and to get a lower bound for $\ln|\det E_{-+}^\delta|$.

9 Lower bounds on the determinant

We keep the assumptions formulated in the beginning of Section 8, in particular (8.1). We restrict the attention to the case when (8.9) holds with M given by (8.6), and recall that so is the case with probability $\geq 1 - Ce^{-h^{-2n}/C}$. The restrictions (8.15), (8.17) on δ will be further strengthened below.

Using a formula of the type (8.25) we shall show that for every $z \in \widetilde{\Omega}$, the determinant of $P_\delta(z)$ is very likely not to be too small. For that we study the probability distribution of the random matrix E^δ_{-+}, and show that we are close enough to the Gaussian case to be able to apply the results of Section 7 to the determinant. Recall that we work under the assumption (8.9), which is fulfilled with probability $\geq 1 - Ce^{-h^{-2n}/C}$. We want to study the map

$$\begin{aligned} Q \mapsto E^\delta_{-+} &= E^0_{-+} + \sum_1^\infty E^0_-(\delta\widetilde{Q}E^0)^{j-1}\delta\widetilde{Q}E^0_+ \\ &= E^0_{-+} + \delta E^0_-\widetilde{Q}E^0_+ + \sum_2^\infty \left(\frac{\delta Ch^{-n}}{\sqrt{\alpha}}\right)^j \frac{1}{\sqrt{\alpha}} R_j, \end{aligned}$$

where $\widetilde{Q} = (\widetilde{P} - z)^{-1}Q$ and $\|R_j\|_{\mathrm{HS}} \leq 1$. Here, we used that $\|E^0_\pm\|$, $\|E^0\| \leq 1/\sqrt{\alpha}$. We can rewrite this further as

$$\begin{aligned} E^\delta_{-+} &= E^0_{-+} + \frac{\delta}{\alpha}(\sqrt{\alpha}E^0_-\widetilde{Q}\sqrt{\alpha}E^0_+ + \frac{Ch^{-n}\delta Ch^{-n}}{\sqrt{\alpha}}\sum_0^\infty \left(\frac{\delta Ch^{-n}}{\sqrt{\alpha}}\right)^j R_{j+2}) \qquad (9.1) \\ &=: E^0_{-+} + \frac{\delta}{\alpha}\widehat{Q}. \end{aligned}$$

We strengthen (8.15), (8.17) to

$$\frac{\delta M^2}{\sqrt{\alpha}} \ll 1, \qquad (9.2)$$

and recall that by (8.6), $M = C_1 h^{-n}$.

Then

$$\frac{\delta C h^{-n}}{\sqrt{\alpha}} \ll \frac{h^n}{C} \ll 1,$$

and we get

$$\widehat{Q} = \sqrt{\alpha}E^0_-\widetilde{Q}\sqrt{\alpha}E^0_+ + T,\ \|T\|_{\mathrm{HS}} \leq \frac{C^2h^{-2n}\delta}{\sqrt{\alpha}} \ll 1. \qquad (9.3)$$

In view of (8.1) we have

$$\sqrt{\alpha}E_-^0\widetilde{Q}\sqrt{\alpha}E_+^0=\sqrt{\alpha}E_-^0(\widetilde{P}-z)^{-1}\widehat{S}\sum_{j,k}\widehat{e}_j\alpha_{j,k}\widetilde{e}_k^*\widetilde{S}\sqrt{\alpha}E_+^0, \tag{9.4}$$

where we recall from (5.8) that

$$\sqrt{\alpha}E_+^0v_+=\sum_1^N v_+(j)e_j,\quad \sqrt{\alpha}E_-^0v(j)=(v|f_j),\quad 1\le j\le N, \tag{9.5}$$

where $e_1,..,e_N$ and $f_1,...,f_N$ are orthonormal bases for $\mathrm{ran}(1_{[0,\alpha]}(P(z)^*P(z)))$ and $\mathrm{ran}(1_{[0,\alpha]}(P(z)P(z)^*))$ respectively, writing $\mathrm{ran}(B)$ for the range of B.

Here, we wish to apply the discussion of Section 13. The operators $\widetilde{S}\sqrt{\alpha}E_+^0$, $\sqrt{\alpha}E_-^0(\widetilde{P}-z)^{-1}\widehat{S}$ are clearly Hilbert-Schmidt of rank $\le N$. Let $\widetilde{t}_j$, $\widehat{t}_j$ denote the singular values of these operators so that $\widetilde{t}_j=\widehat{t}_j=0$ for $j\ge N+1$.

Lemma 9.1 *We have*

$$\frac{1}{C}\le \widetilde{t}_j,\,\widehat{t}_j\le C,\ 1\le j\le N, \tag{9.6}$$

where $C>0$ is independent of h,α.

Proof: (9.5) shows that $\|\sqrt{\alpha}E_+^0\|,\|\sqrt{\alpha}E_-^0\|\le 1$, and clearly $\|(\widetilde{P}-z)^{-1}\widehat{S}\|,\|\widetilde{S}\|=\mathcal{O}(1)$, so the upper bound in (9.6) is clear.

On the other hand, $\sqrt{\alpha}E_+^0v_+$ is confined to a bounded region in phase space, and it is easy to show that

$$C\|\widetilde{S}\sqrt{\alpha}E_+^0v_+\|\ge\|\sqrt{\alpha}E_+^0v_+\|=\|v_+\|,$$

which implies that the smallest eigenvalue of $((\widetilde{S}\sqrt{\alpha}E_+^0)^*(\widetilde{S}\sqrt{\alpha}E_+^0))^{1/2}$ is $\ge 1/C$. The lower bound on $\widetilde{t}_j$ follows. The argument for $\widehat{t}_j$ is essentially the same. $\Box$

Let $\widehat{f}_1,...,\widehat{f}_N$ and $\widetilde{f}_1,...,\widetilde{f}_N$ be orthonormal bases in $\mathbf{C}^N$ of eigenfunctions of $((\sqrt{\alpha}E_-^0(\widetilde{P}-z)^{-1}\widehat{S})(\sqrt{\alpha}E_-^0(\widetilde{P}-z)^{-1}\widehat{S})^*)^{1/2}$ and $((\widetilde{S}\sqrt{\alpha}E_+^0)^*(\widetilde{S}\sqrt{\alpha}E_+^0))^{1/2}$ respectively, with $\widehat{t}_j$ and $\widetilde{t}_k$ as the corresponding eigenvalues. We can then choose the orthonormal bases $\{\widehat{e}_j\}$, $\{\widetilde{e}_j\}$ in L^2 so that

$$\widehat{e}_j=\frac{1}{\widehat{t}_j}(\sqrt{\alpha}E_-^0(\widetilde{P}-z)^{-1}\widehat{S})^*\widehat{f}_j,\quad \widetilde{e}_j=\frac{1}{\widetilde{t}_j}(\widetilde{S}\sqrt{\alpha}E_+^0)\widetilde{f}_j, \tag{9.7}$$

for $j=1,2,...,N$. Then from (9.4), we get

$$\sqrt{\alpha}E_-^0\widetilde{Q}\sqrt{\alpha}E_+^0=\sum_{1\le j,k\le N}\widehat{t}_j\widetilde{t}_k\alpha_{j,k}\widehat{f}_j\widetilde{f}_k^*. \tag{9.8}$$

Now we will be a a little more specific about the assumption (8.9). We will restrict the attention to the set $\mathcal{Q}_M$ of matrices $(\alpha_{j,k}(\omega))$ such that

$$\|\widehat{S}_2 \sum \alpha_{j,k}(\omega)\widehat{e}_j \widetilde{e}_k^* \widetilde{S}\|_{\mathrm{HS}} \leq M, \tag{9.9}$$

which implies (8.9) and which is fulfilled with probability $\geq 1 - C\exp\,(-h^{-2n}/C)$. Here we recall that we assumed $\widehat{m}$ to be integrable and wrote $\widehat{S} = \widehat{S}_1\widehat{S}_2$ with $\widehat{S}_j \in \mathrm{Op}(S(\widehat{m}^{1/2}))$. (When $\widetilde{m}$ is integrable instead, we make a corresponding factorization of $\widetilde{S}$.)

(9.3) can be reformulated as

$$\widehat{Q}(\alpha) = \mathrm{diag}\,(\widehat{t}_j) \circ \Big((\alpha_{j,k})_{1\leq j,k\leq N} + \widetilde{T}(\alpha.)\Big) \circ \mathrm{diag}\,(\widetilde{t}_k), \tag{9.10}$$

$$\|\widetilde{T}(\alpha.)\|_{\mathrm{HS}} \leq \mathcal{O}(1)\frac{\delta M^2}{\sqrt{\alpha}}, \tag{9.11}$$

for $(\alpha_{j,k}) \in \mathcal{Q}_M$.

Let $|||(\alpha_{j,k})|||$ denote the norm in (9.9) and let $\mathcal{H}$ be the corresponding Hilbert space of $\mathbf{N}\times\mathbf{N}$ matrices. We shall view $\mathrm{HS}(\mathbf{C}^N, \mathbf{C}^N) =: \mathcal{H}_N$ as a subspace of $\mathcal{H}$ in the natural way. Note that the two norms are uniformly equivalent on this subspace.

The Cauchy inequality implies (after decreasing M by a constant factor) that the differential of the map $\alpha. \mapsto \widetilde{T}(\alpha.)$ satisfies the following estimate on $\mathcal{Q}_M$:

$$\|d\widetilde{T}\|_{\mathcal{H}\to\mathcal{H}_N} = \mathcal{O}(1)\frac{\delta M}{\sqrt{\alpha}}. \tag{9.12}$$

On $\mathcal{H}$, $\mathcal{H}_N$ we have the basic probability measures,

$$\mu_{\mathcal{H}} = \prod_{j,k=1}^{\infty} (e^{-|\alpha_{j,k}|^2}\frac{L(d\alpha_{j,k})}{\pi}) \quad \mu_{\mathcal{H}_N} = \prod_{j,k=1}^{N} (e^{-|\alpha_{j,k}|^2}\frac{L(d\alpha_{j,k})}{\pi}). \tag{9.13}$$

We shall now estimate $\Pi_*(\mu_{\mathcal{H}})$ on $\mathcal{Q}_M$, where

$$\Pi((\alpha_{j,k})) = (\alpha_{j,k})_{1\leq j,k\leq N} + \widetilde{T}(\alpha.), \tag{9.14}$$

and to do so, we identify $\widetilde{T}(\alpha.)$ with its image in $\mathcal{H}$ under the natural inclusion $\mathcal{H}_N \subset \mathcal{H}$, and write

$$\Pi = \Pi_0 \circ \kappa, \quad \kappa(\alpha.) = \alpha. + \widetilde{T}(\alpha.), \quad \Pi_0(\alpha.) = (\alpha_{j,k})_{1\leq j,k\leq N}, \tag{9.15}$$

for $\alpha_\cdot = (\alpha_{j,k}) \in \mathcal{H}$.

We first proceed formally, ignoring some technical difficulties due to the infinite dimension. We have

$$\begin{aligned} | \, \|\kappa(\alpha_\cdot)\|_{\mathrm{HS}}^2 - \|\alpha_\cdot\|_{\mathrm{HS}}^2 \, | &= | \, \|\Pi_0\kappa(\alpha_\cdot)\|_{\mathrm{HS}}^2 - \|\Pi_0\alpha_\cdot\|_{\mathrm{HS}}^2 \, | \\ &= | \, \|\Pi_0\kappa(\alpha_\cdot)\|_{\mathrm{HS}} - \|\Pi_0\alpha_\cdot\|_{\mathrm{HS}} \, | \times | \, \|\Pi_0\kappa(\alpha_\cdot)\|_{\mathrm{HS}} + \|\Pi_0\alpha_\cdot\|_{\mathrm{HS}} \, | \\ &\leq \|\widetilde{T}(\alpha_\cdot)\|_{\mathrm{HS}}\big(2\|\alpha_\cdot\| + \mathcal{O}(\frac{\delta M^2}{\sqrt{\alpha}})\big) \\ &\leq \mathcal{O}(1)\frac{\delta M^3}{\sqrt{\alpha}}, \end{aligned} \tag{9.16}$$

where we strengthen the assumption (9.2) to

$$\frac{\delta M^3}{\sqrt{\alpha}} \ll 1, \text{ or equivalently } \delta \ll h^{3n+1/2}. \tag{9.17}$$

As for the Jacobian of κ, we recall that if $A : \mathcal{H} \to \mathcal{H}$ is linear with $\|A\|_{\mathrm{tr}} \ll 1$ (uniformly with respect to M), then $\det(1 + A) = 1 + \mathcal{O}(\|A\|_{\mathrm{tr}})$. Also, if A is of rank $\leq N^2$, we know that $\|A\|_{\mathrm{tr}} \leq N^2\|A\|$, so in our case we get from (9.12) that

$$\det \frac{\partial \kappa}{\partial x} = 1 + \mathcal{O}(1)\frac{\delta N^2 M}{\sqrt{\alpha}}.$$

Here the remainder term is $\ll 1$ in view of the assumption (9.17) and the fact that $N \ll M$. (Recall that $N = \mathcal{O}(\alpha^\kappa h^{-n})$ by (4.27).)

If F is a locally defined holomorphic map : $\mathcal{H} \to \mathcal{H}$, then

$$L(dF(x)) = |\det \frac{\partial F}{\partial x}|^2 L(dx),$$

so in our case,

$$L(d\kappa(x)) = (1 + \mathcal{O}(1)\frac{\delta N^2 M}{\sqrt{\alpha}})L(dx).$$

Combining this with (9.16), we get

$$\kappa_*(\mu_{\mathcal{H}}) \leq (1 + \mathcal{O}(1)\frac{\delta M^3}{\sqrt{\alpha}})\mu_{\mathcal{H}} \text{ on } \mathcal{Q}_M.$$

Since $(\Pi_0)_*\mu_{\mathcal{H}} = \mu_{\mathcal{H}_N}$, we conclude that

$$\Pi_*(\mu_{\mathcal{H}}) \leq (1 + \mathcal{O}(1)\frac{\delta M^3}{\sqrt{\alpha}})\mu_{\mathcal{H}_N} \text{ on } \mathcal{Q}_M. \tag{9.18}$$

At the end of this section we shall complete the proof of (9.18) by means of finite dimensional approximations.

For $\alpha. = \alpha.(\omega) \in \mathcal{Q}_M$ we want to estimate the probability that $|\det E_{-+}^\delta|$ is small. According to Proposition 7.3, the $\mu_{\mathcal{H}_N}(d\check{Q})$-measure of the set of matrices $\check{Q}$ with

$$|\det(\mathrm{diag}\,(\widehat{t}_j)^{-1} \circ \frac{\alpha}{\delta} E_{-+}^0 \circ \mathrm{diag}\,(\widetilde{t}_j)^{-1} + \check{Q})| \le e^a$$

is

$$\le Ce^{-\frac{1}{2}(C_2+(N-\frac{1}{2})\ln N-2N-a)},$$

if

$$a \le C_2 + (N+\frac{1}{2})\ln N - 2N. \tag{9.19}$$

In view of (9.1), (9.10), (9.18) this is also (after a slight increase of C) an upper bound for the probability to have $(\alpha_{j,k}) \in \mathcal{Q}_M$ and

$$|\det(\mathrm{diag}\,(\widehat{t}_j)^{-1} \circ (\frac{\alpha}{\delta} E_{-+}^0 + \widehat{Q}) \circ \mathrm{diag}\,(\widetilde{t}_j)^{-1})| \le e^a,$$

or equivalently that

$$|\det E_{-+}^\delta| \le e^a (\frac{\delta}{\alpha})^N \prod_1^N \widehat{t}_j \prod_1^N \widetilde{t}_j.$$

Write

$$a = C_2 + (N-\frac{1}{2})\ln N - 2N - b \tag{9.20}$$

and restrict the attention to $b \ge 0$. Then

$$e^a = e^{C_2+(N-\frac{1}{2})\ln N-2N-b}$$

and we get

$$\mathsf{P}((9.9) \text{ holds and } |\det E_{-+}^\delta| \le e^{N\ln\frac{1}{\alpha}-N\ln\frac{1}{\delta}+(N-\frac{1}{2})\ln N-CN+C_2-b}) \le e^{-b}. \tag{9.21}$$

Summing up the discussion so far, we have

Proposition 9.2 *Consider the Grushin problem (8.19). Assume (4.21) and choose $\alpha = Ch$, $C \gg 0$. Then there exist positive constants $\widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2, \widetilde{C}$ such that for $b \ge 0$*

$$\begin{aligned} &\mathsf{P}(\textit{(9.9) holds and } |\det E_{-+}^\delta| \ge e^{-\widetilde{C}_0 h^{\kappa-n}\ln\frac{1}{h}-\widetilde{C}_1-\widetilde{C}_2 h^{\kappa-n}\ln\frac{1}{\delta}-b}) \\ &\ge 1 - \widetilde{C}e^{-b} - \widetilde{C}e^{-C_0 h^{-2n}}. \end{aligned} \tag{9.22}$$

Here δ is assumed to satisfy (9.17).

In view of (8.24), (8.25), we get

Theorem 9.3 *We now return to the original $P_\delta(z)$ in (8.12) and we assume (4.21) uniformly for all z in some open set $\widehat{\Omega} \Subset \widetilde{\Omega}$. If δ satisfies (8.15), (8.17) there is a constant $C > 0$ such that*

$$\ln|\det P_\delta| \leq \frac{1}{(2\pi h)^n}(\iint \ln|p|dxd\xi + Ch^\kappa \ln\frac{1}{h}),\ \forall z \in \widehat{\Omega}, \tag{9.23}$$

with probability $\geq 1 - Ce^{-C_0 h^{-2n}}$. If δ satisfies the stronger condition (9.17), then there are constants $C, \widetilde{C}, C_0 > 0$ such that for every $z \in \widehat{\Omega}$ and $\epsilon \geq 0$:

$$\ln|\det P_\delta| \geq \frac{1}{(2\pi h)^n}(\iint \ln|p|dxd\xi - Ch^\kappa(\ln\frac{1}{h} + \ln\frac{1}{\delta}) - \epsilon) \tag{9.24}$$

with probability $\geq 1 - e^{-\epsilon(2\pi h)^{-n}} - \widetilde{C}e^{-C_0 h^{-2n}}$.

Notice that the last term in the lower bound for the probability is much smaller than the second term, and can therefore be eliminated.

We end this section by completing the proof of (9.18) by finite dimensional approximation. (We suggest the reader to proceed directly to Section 10.)

Lemma 9.4 *We can choose the orthonormal bases $\{\widehat{e}_j\}$, $\{\widetilde{e}_j\}$ in L^2 so that (9.7) is fulfilled for $1 \leq j \leq N$ and such that the square of the norm in (9.9) is equivalent to*

$$\sum_{j,k} |\alpha_{j,k}|^2 \widehat{\mu}_2(j)^2 \widetilde{\mu}(k)^2, \tag{9.25}$$

where $\widehat{\mu}_2(j)$, $\widetilde{\mu}(k)$ denote the singular values of $\widehat{S}_2$ and $\widetilde{S}$ respectively.

In this lemma we did not try to have any uniformity with respect to h.

Assume the lemma for a while. Then for $\widetilde{N} \geq N+1$, we can replace Q_ω in (8.1) by

$$Q_\omega^{\widetilde{N}} = \widehat{S} \circ \Big(\sum_{1 \leq j,k \leq \widetilde{N}} \alpha_{j,k}(\omega) \widehat{e}_j \widetilde{e}_k^* + \sum_{j \text{ or } k \geq \widetilde{N}+1} \beta_{j,k}^{\widetilde{N}}(\alpha^{\widetilde{N}}(\omega)) \widehat{e}_j \widetilde{e}_k^* \Big) \circ \widetilde{S},$$

which depends on finitely many random variables. Here $\alpha^{\widetilde{N}}(\omega) = (\alpha_{j,k}(\omega))_{1 \leq j,k \leq \widetilde{N}}$ and $\beta_{j,k}^{\widetilde{N}}$ are the linear functions of $\alpha^{\widetilde{N}}$ which minimize

$$\|\widehat{S}_2 \circ (\sum_{j,k \leq \widetilde{N}} \alpha_{j,k} \widehat{e}_j \widetilde{e}_k^* + \sum_{j \text{ or } k > \widetilde{N}} \beta_{j,k} \widehat{e}_j \widetilde{e}_k^*) \circ \widetilde{S}\|_{\mathrm{HS}}.$$

Here we can use the $\widehat{e}_j$, $\widetilde{e}_j$ of Lemma 9.4. On the set $\mathcal{Q}_M$, we have $\widehat{S}_2 \circ Q^{\widetilde{N}} \circ \widetilde{S} \to \widehat{S}_2 Q \widetilde{S}$ in Hilbert-Schmidt norm, and $\|\widehat{S}_2 Q^{\widetilde{N}} \widetilde{S}\|_{\mathrm{HS}} \le M$ when $\alpha(\omega) \in \mathcal{Q}_M$.

We get the corresponding matrix $E_{-+}^{\delta,\widetilde{N}} = E_{-+}^0 + \frac{\delta}{\alpha} \widehat{Q}_{\widetilde{N}}$ and $\widehat{Q}_{\widetilde{N}}$ can be written as in (9.10) with $\widetilde{T}(\alpha)$ replaced by $\widetilde{T}_{\widetilde{N}}(\alpha)$ satisfying (9.11). Now instead of $\mu_{\mathcal{H}}$ we have the finite dimensional measure

$$\mu_{\mathcal{H}_{\widetilde{N}}} = \prod_{j,k=1}^{\widetilde{N}} (e^{-|\alpha_{j,k}|^2} \frac{L(d\alpha_{j,k})}{\pi}),$$

which we can view as the restriction of $\mu_{\mathcal{H}}$ to the tribe generated by $\alpha_{j,k}$ with $1 \le j,k \le \widetilde{N}$) and we define $\Pi^{\widetilde{N}}$ as in (9.14) with $\widetilde{T}$ replaced by $\widetilde{T}_{\widetilde{N}}$. The subsequent arguments now become rigorous since we are in finite dimension and we get

$$\Pi_*^{\widetilde{N}}(\mu_{\mathcal{H}_{\widetilde{N}}}) \le (1 + \mathcal{O}(1) \frac{\delta M^3}{\sqrt{\alpha}}) \mu_{\mathcal{H}_N} \text{ on } \mathcal{Q}_M.$$

Since $\Pi^{\widetilde{N}} \to \Pi$ on $\mathcal{Q}_M$, we obtain (9.18) in the limit.

We next prove Lemma 9.4.

Proof: We consider first the following simplified problem. Let $m_S \le 1$ be a square integrable order function and let $S \in \mathrm{Op}\,(S(m_S))$ be elliptic. We look for an orthonormal basis $e_1, e_2, ..$ in L^2 such that

$$\| \sum u_k S e_k \|^2 \backsim \sum \mu_j(S)^2 |u_k|^2, \tag{9.26}$$

where $\mu_1(S) \ge \mu_2(S) \ge ... \to 0$ are the singular values of S and such that

$$e_1, ..., e_{N_0}$$

is a prescribed orthonormal family of functions in $\mathcal{S}$.

Since $\sum \mu_j^2 = \mathcal{O}(h^{-n})$, we have $N\mu_N^2 = \mathcal{O}(h^{-n})$, and using also that $\mu_N \le \mathcal{O}(1)$, we get

$$\mu_N \le \frac{\mathcal{O}(1)}{(Nh^n)^{1/2} + 1}.$$

On the other hand there exists a constant $\kappa_0 > 0$ such that

$$m_S(\rho) \ge \frac{1}{C_0} \langle \rho \rangle^{-\kappa_0},$$

and we can use the mini-max principle to compare the eigenvalues of $(S^*S)^{1/2}$ with those of $(1 + ((hD)^2 + x^2))^{-\kappa_0/2}$ and deduce that

$$\mu_N \ge \frac{1}{\mathcal{O}(1)} \frac{1}{(1 + hN^{1/n})^{\kappa_0/2}}.$$

If 0< $\mu \ll 1$, we have, with p denoting the symbol of $(S^*S)^{1/2}$, that

$$\begin{aligned}
\mathrm{dist}\,(0, p^{-1}([0, 2\mu])) &\geq \mathrm{dist}\,(0, m_S^{-1}([0, \frac{2\mu}{C}])) \\
&\geq \mathrm{dist}\,(0, \{\rho; \frac{1}{C_0}\langle\rho\rangle^{-\kappa_0} \leq \frac{2\mu}{C}\}) \\
&\geq \frac{1}{C_1}\mu^{-1/\kappa_0}.
\end{aligned}$$

If u is a corresponding normalized eigenfunction, we have $(\mu^{-1}(S^*S)^{1/2}-1)u = 0$, and we notice that $\mu^{-1}(S^*S)^{1/2} \in \mathrm{Op}\,(S(\mu^{-1}m_S))$, where $\mu^{-1}m_S$ satisfies uniformly the axioms of an order function, when $\mu \to 0$. We conclude that

$$u = \mathcal{O}(1), \text{ in } H(m)$$

uniformly with respect to m if $m = m_\mu$ belongs to a family of orderfunctions that satisfy uniformly the axioms and $m = 1$ on $\{\rho \in T^*\mathbf{R}^n;\, p(\rho) \leq 2\mu\}$. From this, we deduce that

$$(\varphi|u) = \mathcal{O}(\mu^N),\ \forall N,$$

if $\varphi \in \mathcal{S}$ is fixed, and $\mu \to 0$.

Let $f_1, f_2, ...$ be an orthonormal basis of eigenfunctions of $(S^*S)^{1/2}$ with $\mu_1 \geq \mu_2 \geq ...$ the corresponding decreasing enumeration of eigenvalues. Then $(e_j|f_k) = \mathcal{O}(k^{-\infty})$, $1 \leq j \leq N_0$, $k \geq 1$. Let $N \gg N_0$. For $j \geq N+1$, put

$$g_j = f_j - \Pi_{E_{N_0}} f_j = f_j + r_j,\ E_{N_0} \ni r_j = \mathcal{O}(j^{-\infty}). \tag{9.27}$$

Here, we let $E_{N_0} = (e_1, ..., e_{N_0})$ be the span of $e_1, ..., e_{N_0}$ and $\Pi_{E_{N_0}} : L^2 \to E_{N_0}$ be the corresponding orthogonal projection. Then $g_j \in E_{N_0}^\perp$ and for $j, k > N$:

$$(g_j|g_k) = \delta_{j,k} + \mathcal{O}(j^{-\infty}k^{-\infty}). \tag{9.28}$$

Here the estimates are uniform with respect to N and if N is sufficiently large, we see that $G = ((g_j|g_k))$ is a positive definite matrix of which any real power has elements satisfying (9.28). Let $(a_{j,k}) = G^{-1/2}$, so that $a_{j,k} = \delta_{j,k} + \mathcal{O}(j^{-\infty}k^{-\infty})$, $j, k > N$. Put

$$e_j = \sum_{k>N} a_{j,k} g_k,\ j > N.$$

Then e_j, $j > N$ form an orthonormal basis in the span $G_N^\perp$ of $g_{N+1}, g_{N+2}, ...$ We see that for $j > N$:

$$e_j = f_j + \sum_{k>N} \mathcal{O}(j^{-\infty}k^{-\infty}) f_k + \widetilde{r}_j,\ E_{N_0} \ni \widetilde{r}_j = \mathcal{O}(j^{-\infty}). \tag{9.29}$$

$G_N = (G_N^\perp)^\perp$ is a space of dimension N, containing E_{N_0}. For $1 \le j \le N$, we consider

$$\Pi_{G_N} f_j = f_j - \Pi_{G_N^\perp} f_j,$$

$$\begin{aligned}
\Pi_{G_N^\perp} f_j &= \sum_{N+1}^{\infty} (f_j|e_k)e_k \\
&= \sum_{N+1}^{\infty} (f_j|\widetilde{r}_k)e_k \\
&= \sum_{N+1}^{\infty} \mathcal{O}(k^{-\infty})(f_k + \sum_{\ell=N+1}^{\infty} \mathcal{O}(k^{-\infty}\ell^{-\infty})f_\ell + \widetilde{r}_k) \\
&= \sum_{N+1}^{\infty} \mathcal{O}(k^{-\infty})f_k + \mathcal{O}(N^{-\infty}),
\end{aligned}$$

where the last term is in E_{N_0}. Thus, we get for $1 \le j \le N$:

$$\Pi_{G_N} f_j = f_j + \sum_{N+1}^{\infty} \mathcal{O}(k^{-\infty})f_k + \widehat{r}_j, \ E_{N_0} \ni \widehat{r}_j = \mathcal{O}(N^{-\infty}).$$

This implies

$$\Pi_{G_N} f_j = f_j + \sum_{1}^{\infty} \mathcal{O}((k+N)^{-\infty})f_k, \ 1 \le j \le N. \tag{9.30}$$

Now, complete $e_1, ..., e_{N_0}$ to an orthonormal basis $e_1, ..., e_N$ in G_N. Then $e_1, e_2, ...$ is an orthonormal basis in L^2. (9.30) shows that $\Pi_{G_N} f_1, ..., \Pi_{G_N} f_N$ is very close to being an orthonormal basis in G_N, and we see that

$$e_j = \sum_{k=1}^{N} u_{j,k} f_k + \sum_{1}^{\infty} \mathcal{O}((k+N)^{-\infty})f_k, \ 1 \le j \le N, \tag{9.31}$$

where $(u_{j,k})_{1 \le j,k \le N}$ is a unitary matrix. (9.29), (9.31) imply that for all $j \ge 1$:

$$e_j = \sum_{k=1}^{\infty} a_{j,k} f_k + \sum_{k=1}^{\infty} \mathcal{O}((j+N)^{-\infty}(k+N)^{-\infty})f_k, \tag{9.32}$$

where $a_{j,k} = u_{j,k}$ for $j, k \le N$ and $a_{j,k} = \delta_{j,k}$ when $\max(j,k) > N$.

We now fix N sufficiently large so that the above estimates hold. Using (9.32), we get

$$e_j = f_j + \sum_k \mathcal{O}(j^{-\infty}k^{-\infty})f_k,\ \ Se_j = \mu_j f_j + \sum_k \mathcal{O}(j^{-\infty}k^{-\infty})f_k,$$

$$(Se_k|Se_j) = \mu_k^2\delta_{j,k} + \mathcal{O}(k^{-\infty}j^{-\infty}),$$

$$\begin{aligned}\|\sum_1^\infty u_k Se_k\|^2 &= \sum_{k=1}^\infty \mu_k^2 u_k\overline{u}_k + \sum_{k=1}^\infty\sum_{j=1}^\infty \mathcal{O}(k^{-\infty}j^{-\infty})u_k\overline{u}_j \\ &= ((\mu^2+K)u|u)_{\ell^2} \\ &= ((1+\mu^{-1}K\mu^{-1})\mu u|\mu u)\end{aligned}$$

where μ denotes the operator $\mathrm{diag}\,(\mu_j)$. Here $\mu^{-1}K\mu^{-1}$ is compact: $\ell^2\to\ell^2$, so $1+\mu^{-1}K\mu^{-1}$ is a non-negative selfadjoint Fredholm operator of index 0. If $u(j) = \mathcal{O}(j^{M_0})$ and $(1+\mu^{-1}K\mu^{-1})u = 0$, then $u = \mathcal{O}(j^{-\infty})$. If $0 \neq v \in \ell^2$, $(1+\mu^{-1}K\mu^{-1})v = 0$, we conclude that

$$((1+\mu^{-1}K\mu^{-1})v|v) = ((\mu^2+K)u|u) = \|S(\sum_1^\infty u_k e_k)\|^2 > 0$$

thus $1+\mu^{-1}K\mu^{-1}$ is bijective and we finally conclude that (9.26) holds.

Now we can finish the proof of the lemma. We choose $\{\widehat{e}_j\}$, $\{\widetilde{e}_j\}$ in (9.7) so that (9.26) holds with $S=\widehat{S}_2$, $e_j=\widehat{e}_j$ and $S=\widetilde{S}^*$, $e_j=\widetilde{e}_j$ respectively. Then the square of the norm (9.9) is equal to

$$\sum_{i,j,k,\ell} (\widehat{S}_2\widehat{e}_i|\widehat{S}_2\widehat{e}_j)(\widetilde{S}^*\widetilde{e}_k|\widetilde{S}^*\widetilde{e}_\ell)\alpha_{i,k}\overline{\alpha}_{j,\ell} = (\widehat{\mathcal{S}}\otimes\widetilde{\mathcal{S}}\alpha|\alpha)_{\ell^2\otimes\ell^2}, \tag{9.33}$$

where

$$\widehat{\mathcal{S}}_{j,i} = (\widehat{S}_2\widehat{e}_i|\widehat{S}_2\widehat{e}_j),\ \ \widetilde{\mathcal{S}}_{\ell,k} = (\widetilde{S}^*\widetilde{e}_k|\widetilde{S}^*\widetilde{e}_\ell).$$

From (9.26) we know that

$$\begin{aligned}\widehat{\mathcal{S}} &= \widehat{\mu}\widehat{P}\widehat{P}\widehat{\mu},\ \ \widehat{\mu} = \mathrm{diag}\,(\widehat{\mu}_2(j)) \\ \widetilde{\mathcal{S}} &= \widetilde{\mu}\widetilde{P}\widetilde{P}\widetilde{\mu},\ \ \widetilde{\mu} = \mathrm{diag}\,(\widetilde{\mu}(j)),\end{aligned}$$

where $\widehat{P}$, $\widetilde{P}$ are positive selfadjoint operators satisfying

$$\frac{1}{C}I \le \widehat{P},\, \widetilde{P} \le CI.$$

Then (9.33) can be written

$$\|(\widehat{P}\widehat{\mu} \otimes \widetilde{P}\widetilde{\mu})\alpha\|^2_{\ell^2\otimes\ell^2},$$

and the lemma follows. $\Box$

10 Spectral asymptotics when dp, $d\overline{p}$ are independent

Let $\Gamma \Subset \Omega$ be open with C^2 boundary and assume that for every $z \in \partial\Gamma$:

$$\Sigma_z := p^{-1}(z) \text{ is a smooth sub-manifold of } T^*\mathbf{R}^n \text{ on which } dp, d\overline{p} \text{ are linearly independent at every point.} \tag{10.1}$$

This assumption, which is satisfied also in a neighborhood of $\partial\Gamma$, implies that $\operatorname{codim}(\Sigma_z) = 2$. The assumption can also be rephrased more briefly by saying that $\partial\Gamma$ does not contain any critical value of $p : \mathbb{R}^{2n} \to \mathbb{R}^2$. Here p is the leading symbol of the original (z-independent operator.) If $p_z(\rho) = (\widetilde{p}(\rho) - z)^{-1}(p(\rho) - z)$ is the principal symbol of $(\widetilde{P} - z)^{-1}(P - z)$, we introduce

$$I(z) = \int_{\mathbf{R}^2} \ln|p_z(\rho)|d\rho \tag{10.2}$$

which is the same integral as in (9.23), (9.24) (where z was fixed). It is easy to see that $I(z)$ is a smooth function on the neighborhood of $\partial\Gamma$ where (10.1) holds and as in [10] we can compute $\Delta_z I(z)$. Since $z \mapsto p_z(\rho)$ is holomorphic, we know that $\Delta_z \ln|p_z(\rho)| = 0$ when $p_z(\rho) \neq 0$, ie when $\rho \notin \Sigma_z$. On the other hand $p_z(\rho) = (\widetilde{p}(\rho) - z)^{-1}(p(\rho) - z)$ where the first factor is holomorphic in z and non-vanishing, so

$$\Delta_z \ln|p_z(\rho)| = \Delta_z \ln|p(\rho) - z| = 2\pi\delta(z - p(\rho)).$$

If $\varphi \in C_0^\infty(\Omega)$, we get

$$\begin{aligned}\int(\Delta_z I(z))\varphi(z)L(dz) &= \int\int \Delta_z(\ln|p_z(\rho)|)\varphi(z)L(dz)d\rho\\ &= 2\pi\int\int \delta(z-p(\rho))\varphi(z)L(dz)d\rho = 2\pi\int\varphi(p(\rho))d\rho.\end{aligned}$$

Thus we get (as in [6, 7] when $n = 1$):

$$\frac{1}{2\pi}\Delta(I(z))L(dz) = p_*(d\rho) \text{ near } \partial\Gamma, \tag{10.3}$$

where $d\rho$ is the symplectic volume element. Notice that this formula is still true without the assumption (10.1) and hence not only in a neighborhood of $\partial\Gamma$, but in Ω; however $I(z)$ is no more smooth in general but still well-defined as a distribution. This fact will be used in the proof of Theorem 10.1.

In view of (10.1), we have $V(t) \backsim t$ and (4.21) holds uniformly with $\kappa = 1$, when z varies in a neighborhood of $\partial\Gamma$. Correspondingly the conclusions in Theorem 9.3 hold uniformly, when z varies in a small neighborhood of $\partial\Gamma$.

Theorem 10.1 *Let $\Gamma \Subset \Omega$ be open with C^2 boundary and make the assumption (10.1). Let $\delta > 0$ satisfy (9.17) and assume that $h \ln \frac{1}{\delta} \ll \epsilon \ll 1$ (or equivalently $\delta \geq e^{-\epsilon/(Ch)}$, $C \gg 1$, $\epsilon \ll 1$, implying also that $\epsilon \geq \widetilde{C} h \ln \frac{1}{h}$ for some $\widetilde{C} > 0$). Then with $C > 0$ large enough, the number $N(P_\delta, \Gamma)$ of eigenvalues of P_δ in Γ satisfies*

$$|N(P_\delta, \Gamma) - \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma))| \leq C \frac{\sqrt{\epsilon}}{h^n} \tag{10.4}$$

with probability

$$\geq 1 - \frac{C}{\sqrt{\epsilon}} e^{-\frac{\epsilon/2}{(2\pi h)^n}}.$$

Proof: The eigenvalues of P_δ in $\widetilde{\Omega}$ coincide with the zeros of the holomorphic function

$$F_\delta(z) = \det P_\delta(z). \tag{10.5}$$

Theorem 9.3 tells us that there exists a neighborhood $\widehat{\Omega}$ of $\partial\Omega$ such that

(a) With probability $\geq 1 - Ce^{-C_0 h^{-2n}}$, we have

$$\ln |F_\delta(z)| \leq \frac{1}{(2\pi h)^n} (I(z) + Ch \ln \frac{1}{h}),\ z \in \widehat{\Omega}.$$

(b) For every $z \in \widehat{\Omega}$ and $\epsilon > 0$ we have

$$\ln |F_\delta(z)| \geq \frac{1}{(2\pi h)^n} (I(z) - Ch(\ln \frac{1}{h} + \ln \frac{1}{\delta}) - \epsilon),$$

with probability $\geq 1 - e^{-\epsilon (2\pi h)^{-n}} - Ce^{-C_0 h^{-2n}}$. Notice here that $\ln \frac{1}{\delta} \geq \ln \frac{1}{h}$.

We can then repeat the arguments of [6, 7]. Recall Proposition 6.1 from [7] proved in a more general form in [6].

Proposition 10.2 *Let $\widehat{\Omega}$, $\widetilde{\Omega}$ be open neighborhoods of $\partial\Gamma$ and $\overline{\Gamma}$ respectively. Let $\varphi \in C^\infty(\widehat{\Omega};\mathbf{R})$ and let f be a holomorphic function in $\widetilde{\Omega}$ such that*

$$|f(z;\widetilde{h})| \le e^{\varphi(z)/\widetilde{h}},\ \ z \in \widehat{\Omega},\, 0 < \widetilde{h} \ll 1. \tag{10.6}$$

Assume that for some $\epsilon > 0$, $\epsilon \ll 1$, $\exists z_k \in \widehat{\Omega}$, $k \in J$, such that

$$\partial\Gamma \subset \bigcup_{k\in J} D(z_k, \sqrt{\epsilon}),\ \ \#J = \mathcal{O}(\frac{1}{\sqrt{\epsilon}}),$$

$$|f(z_k;\widetilde{h})| \ge e^{\frac{1}{\widetilde{h}}(\varphi(z_k)-\epsilon)},\ \ k \in J. \tag{10.7}$$

Then the number of zeros of f in Γ satisfies

$$\#(f^{-1}(0)\cap\Gamma) = \frac{1}{2\pi\widetilde{h}}\int_\Gamma \Delta\varphi L(dz) + \mathcal{O}(\frac{\sqrt{\epsilon}}{\widetilde{h}}),$$

where we let φ denote some distribution in $\mathcal{D}'(\Gamma\cup\widehat{\Omega})$ extending the previous function φ.

The original statement in [6, 7] was with a smooth function φ defined in a whole neighborhood of $\overline{\Gamma}$ satisfying (10.6) there, but the proof works without any changes under the weaker assumptions above.

In view of (a), (b), we can apply the proposition with $\widetilde{h} = (2\pi h)^n$ and ϵ replaced by 2ϵ, $\varphi = I(z) + Ch\ln\frac{1}{h}$, $f = F_\delta$. Then (10.6) holds with a probability as in (a), while (10.7) holds with a probability

$$\ge 1 - \frac{C}{\sqrt{\epsilon}}e^{-\frac{\epsilon}{2}(2\pi h)^{-n}} - Ce^{-C_0h^{-2n}}.$$

We can define φ as a distribution in a full neighborhood of $\overline{\Gamma}$ by (10.2). Then

$$\frac{1}{2\pi\widetilde{h}}\int_\Gamma \Delta\varphi L(dz) = \frac{1}{(2\pi h)^n}\int_\Gamma \frac{1}{2\pi}\Delta I(z)L(dz) = \frac{1}{(2\pi h)^n}\iint_{p^{-1}(\Gamma)} dxd\xi.$$

The theorem follows. $\square$

We next give a result about the simultaneous Weyl asymptotics for a family of domains

Theorem 10.3 *Let $\mathcal{G}$ be a family of domains $\Gamma \Subset \Omega$ that satisfy the assumptions of Theorem 10.1 uniformly in the following sense: Each Γ is of the form $g(z) < 0$ (with $g = g_\Gamma$) where g belongs to a bounded set in $C^2(\overline{\Omega})$ and $g > 1/C$ on $\partial\Omega$ and $|dg| > 1/C$ on $\partial\Gamma$, where $C > 0$ is independent of Γ. We also assume that (10.1) holds for all $z \in \partial\Gamma$, $\Gamma \in \mathcal{G}$, uniformly with respect to (z, Γ).*

Choose δ, ϵ as in Theorem 10.1. Then with probability

$$\geq 1 - \frac{C}{\epsilon} e^{-\frac{\epsilon/2}{(2\pi h)^n}},$$

we have (10.4) with a constant C independent of Γ.

Proof: As in the proof of Theorem 10.1, we use Proposition 10.3 with an appropriate grid of points z_k (see [7] for further details). We now need $\mathcal{O}(1/\epsilon)$ points to achieve that the union of the $D(z_k, \sqrt{\epsilon})$ covers the union of all the $\partial\Gamma$, rather than $\mathcal{O}(1/\sqrt{\epsilon})$ points as in the proof of Theorem 10.1. □

11 Counting zeros of holomorphic functions

Let $\Gamma \Subset \mathbf{C}$ have smooth boundary $\partial\Gamma$. Assume for simplicity that $\gamma := \partial\Gamma$ is connected (or equivalently that Γ is simply connected). This is for notational convenience only. For $0 < r \ll 1$, we put

$$\gamma_r = \gamma + D(0, r) = \partial\Gamma + D(0, r). \tag{11.1}$$

Then γ_r has smooth boundary and is a thin domain of width $\approx 2r$. Let $G_r(z, w)$, $P_r(z, w)$ denote the Green and Poisson kernels of γ_r, so that the Dirichlet problem

$$\Delta u = v,\ u_{|\partial\gamma_r} = f, \quad u, v \in C^\infty(\overline{\gamma_r}),\ f \in C^\infty(\partial\gamma_r),$$

has the unique solution

$$u(z) = \int_{\gamma_r} G_r(z, w) v(w) L(dw) + \int_{\partial\gamma_r} P_r(z, w) f(w) |dw|.$$

We recall some properties of the Green kernel: If $\Omega \Subset \mathbf{C}$ has a smooth boundary and $G_\Omega(x, y)$is the corresponding Green kernel, then

$$G_\Omega \leq 0, \tag{11.2}$$

$$G_\Omega \text{ is } C^\infty \text{ for } x \neq y, \tag{11.3}$$

$$G_\Omega(x,y)=\frac{1+o(1)}{2\pi}\ln|x-y| \text{ for } x\approx y,\ x,y\notin\partial\Omega,\ x-y\to 0. \tag{11.4}$$

$$G_\Omega(\frac{x}{r},\frac{y}{r})=G_{r\Omega}(x,y),\ x,y\in r\Omega. \tag{11.5}$$

$\Omega=\frac{1}{r}\gamma_r$ is a very long domain of approximately constant width and (11.4) is valid uniformly for $x,y\in\Omega$, $|x-y|\le\mathcal{O}(1)$, $\mathrm{dist}\,(x,\partial\Omega),\mathrm{dist}\,(y,\partial\Omega)\ge 1/\mathcal{O}(1)$. Moreover,

$$|G_{r^{-1}\gamma_r}(x,y)|\le C_0e^{-|x-y|/C_0},\ x,y\in r^{-1}\gamma_r,\ |x-y|\ge\frac{1}{\mathcal{O}(1)}. \tag{11.6}$$

To recover these well-known facts, notice that $r^{-1}\gamma_r$ is given by $-1<\varphi(x)<1$, where $\varphi(x)$ is the suitably signed distance from $r^{-1}\partial\Gamma$ to x, so that $|\nabla\varphi(x)|=1$, $|\nabla^2\varphi(x)|=\mathcal{O}(r)$. If $u\in H_0^1(r^{-1}\gamma_r)$, we have by integration by parts,

$$\int_{r^{-1}\gamma_r}((\nabla\varphi)^2+\varphi(x)\Delta\varphi)|u|^2dx=-2\mathrm{Re}\int_{r^{-1}\gamma_r}\varphi(\nabla\varphi\cdot\nabla u)\overline{u}dx,$$

implying

$$\int(1-\mathcal{O}(r))|u|^2dx\le(2+\mathcal{O}(r))\|\nabla u\|\|u\|,$$

$$\|u\|\le(2+\mathcal{O}(r))\|\nabla u\|,$$

$$-\Delta\ge(\frac{1}{4}-\mathcal{O}(r)),$$

where $\Delta=\Delta_{r^{-1}\gamma_r}$ is the Dirichlet Laplacian on $r^{-1}\gamma_r$. From this estimate we can develop exponential decay estimates for $-\Delta$, since we still have a positive lower bound for $\mathrm{Re}\,(e^{\psi}(-\Delta)e^{-\psi})=-\Delta-|\nabla\psi|^2$, if $|\nabla\psi(x)|^2\le 1/5$. We drop the ensuing routine arguments.

In view of (11.5), (11.6) we get

$$|G_r(x,y)|\le C_0e^{-\frac{1}{C_0r}|x-y|},\ x,y\in\gamma_r,\ |x-y|\ge r/\mathcal{O}(1), \tag{11.7}$$

$$|G_r(x,y)|=\frac{1+o(1)}{2\pi}\ln|\frac{x}{r}-\frac{y}{r}|,\ \text{for } |x-y|\le r/C_0, \tag{11.8}$$
$$\mathrm{dist}\,(x,\partial\Omega),\mathrm{dist}\,(y,\partial\Omega)\ge r/C_0,\ |\frac{x}{r}-\frac{y}{r}|\to 0.$$

Let φ be a continuous subharmonic function defined in some neighborhood of $\overline{\gamma_r}$. Let

$$\mu=\mu_\varphi=\Delta\varphi \tag{11.9}$$

be the corresponding locally finite positive measure.

Let u be a holomorphic function defined in a neighborhood of $\overline{\gamma_r}$. We assume that

$$h\ln|u(z)|\le \varphi(z),\ \ z\in\overline{\gamma_r}. \tag{11.10}$$

Lemma 11.1 *Let $C_1, C_2 > 1$ and let $z_0\in\overline{\gamma_{(1-\frac{1}{C_1})r}}$ be a point where*

$$h\ln|u(z_0)|\ge \varphi(z_0)-\epsilon,\ \ 0<\epsilon\ll 1. \tag{11.11}$$

Then the number of zeros of u in $D(z_0,C_2r)\cap\gamma_{(1-\frac{1}{C_2})r}$ is

$$\le\frac{C_3}{h}(\epsilon+\int_{\gamma_r}-G_r(z_0,w)\mu(dw)), \tag{11.12}$$

where C_3 is independent of ϵ, h.

Proof: Writing φ as a uniform limit of an increasing sequence of smooth functions, we may assume that $\varphi\in C^\infty$. Let

$$n_u(dz)=\sum 2\pi\delta(z-z_j),$$

where z_j are the zeros of u counted with their multiplicity. We may assume that no z_j are situated on $\partial\gamma_r$. Then, since $\Delta\ln|u|=n_u$,

$$\begin{aligned} h\ln|u(z)| &= \int_{\gamma_r}G_r(z,w)hn_u(dw)+\int_{\partial\gamma_r}P_r(z,w)h\ln|u(w)||dw| \\ &\le \int_{\gamma_r}G_r(z,w)hn_u(dw)+\int_{\partial\gamma_r}P_r(z,w)\varphi(w)|dw| \\ &= \int_{\gamma_r}G_r(z,w)hn_u(dw)+\varphi(z)-\int_{\gamma_r}G_r(z,w)\mu(dw). \end{aligned} \tag{11.13}$$

Putting $z=z_0$ in (11.13) and using (11.11), we get

$$\int_{\gamma_r}-G_r(z_0,w)hn_u(dw)\le\epsilon+\int_{\gamma_r}-G_r(z_0,w)\mu(dw).$$

Now

$$-G_r(z_0,w)\ge\frac{1}{2\pi C_3},\ \ C_3>0,$$

in $D(z_0,C_2r)\cap\gamma_{(1-\frac{1}{C_2})r}$, and we get (11.12). $\square$

Notice that this argument is basically the same as when using Jensen's formula to estimate the number of zeros of a holomorphic function in a disc. We could assume the bound $h\ln|u(z)|\le\varphi(z)$, in $D(z_0,\widetilde{C}_2r)\cap\gamma_{(1-\frac{1}{\widetilde{C}_2})r}=:\widetilde{\Omega}_r$ for some $\widetilde{C}_2>C_2$. Then we can replace the bound (11.12) by

$$\frac{C_3}{h}(\epsilon+\int_{\widetilde{\Omega}_r}-G_{\widetilde{\Omega}_r}(z_0,w)\mu(dw)),$$

which is sharper, since $-G_{\Omega_1}\le -G_{\Omega_2}$, when $\Omega_1\subset\Omega_2$.

Now we sharpen the assumption (11.11) and assume

$$h\ln|u(z_j)|\ge\varphi(z_j)-\epsilon, \tag{11.14}$$

where $z_1,...,z_N\in\gamma_{(1-\frac{1}{C_1})r}$ are points such that

$$\gamma_r\subset\bigcup_1^N D(z_j,C_1r),\quad N\backsim\frac{1}{r}. \tag{11.15}$$

We may assume that $z_1,z_2,...,z_N$ are arranged in such a way that

$$|z_j-z_k|\backsim r\mathrm{dist}\,(j,k),\ j\ne k, \tag{11.16}$$

where j,k are viewed as elements of $\mathbf{Z}/N\mathbf{Z}$ and we take the natural distance on that set. We will also assume for a while that φ is smooth.

According to Lemma 11.1, we have

$$\#(u^{-1}(0)\cap(D(z_j,C_1r)\cap\gamma_{(1-\frac{1}{C_1})r}))\le\frac{C_3}{h}(\epsilon+\int_{\gamma_r}-G_r(z_j,w)\mu(dw)). \tag{11.17}$$

We introduce

$$\widetilde{r}=(1-\frac{1}{C_1})r \tag{11.18}$$

and consider the harmonic functions on $\gamma_{\widetilde{r}}$,

$$\Psi(z)=h(\ln|u(z)|+\int_{\gamma_{\widetilde{r}}}-G_{\widetilde{r}}(z,w)n_u(dw)), \tag{11.19}$$

$$\Phi(z)=\varphi(z)+\int_{\gamma_{\widetilde{r}}}-G_{\widetilde{r}}(z,w)\mu(dw). \tag{11.20}$$

Then $\Phi(z)\ge\varphi(z)$ with equality on $\partial\gamma_{\widetilde{r}}$. Similarly, $\Psi(z)\ge h\ln|u(z)|$ with equality on $\partial\gamma_{\widetilde{r}}$.

Consider the harmonic function

$$H(z)=\Phi(z)-\Psi(z),\ \ z\in\gamma_{\widetilde{r}}. \tag{11.21}$$

Then on $\partial\gamma_{\widetilde{r}}$, we have by (11.10) that

$$H(z)=\varphi(z)-h\ln|u(z)|\geq 0,$$

so by the maximum principle,

$$H(z)\geq 0,\ \text{ on } \gamma_{\widetilde{r}}. \tag{11.22}$$

By (11.14), we have

$$\begin{aligned}
H(z_j) &= \Phi(z_j)-\Psi(z_j) \qquad (11.23)\\
&= \varphi(z_j)-h\ln|u(z_j)|+\int_{\gamma_{\widetilde{r}}}-G_{\widetilde{r}}(z_j,w)\mu(dw)-\int_{\gamma_{\widetilde{r}}}-G_{\widetilde{r}}(z_j,w)hn_u(dw)\\
&\leq \epsilon+\int_{\gamma_{\widetilde{r}}}-G_{\widetilde{r}}(z_j,w)\mu(dw).
\end{aligned}$$

Harnack's inequality implies that

$$H(z)\leq\mathcal{O}(1)(\epsilon+\int-G_{\widetilde{r}}(z_j,w)\mu(dw))\text{ on }D(z_j,C_1r)\cap\gamma_{(1-\frac{1}{C_1})\widetilde{r}}. \tag{11.24}$$

Now assume that u extends to a holomorphic function in a neighborhood of $\Gamma\cup\overline{\gamma_r}$. We then would like to evaluate the number of zeros of u in Γ. Using (11.17), we first have

$$\#(u^{-1}(0)\cap\gamma_{\widetilde{r}})\leq\frac{C}{h}(N\epsilon+\sum_{j=1}^{N}\int_{\gamma_r}(-G_r(z_j,w))\mu(dw)). \tag{11.25}$$

Let $\chi\in C_0^\infty(\Gamma\cup\gamma_{(1-\frac{1}{C_1})\widetilde{r}};[0,1])$ be equal to 1 on Γ. Of course χ will have to depend on r but we may assume that for all $k\in\mathbf{N}$, and as $r\to 0$,

$$\nabla^k\chi=\mathcal{O}(r^{-k}). \tag{11.26}$$

We are interested in

$$\int\chi(z)hn_u(dz)=\int_{\gamma_{\widehat{r}}}h\ln|u(z)|\Delta\chi(z)L(dz),\ \ \widehat{r}=(1-\frac{1}{C_1})\widetilde{r}. \tag{11.27}$$

Here we have on $\gamma_{\widetilde{r}}$

$$\begin{aligned}
h\ln|u(z)| &= \Psi(z) - \int_{\gamma_{\widetilde{r}}} -G_{\widetilde{r}}(z,w)hn_u(dw) \qquad (11.28)\\
&= \Phi(z) - H(z) - \int_{\gamma_{\widetilde{r}}} -G_{\widetilde{r}}(z,w)hn_u(dw)\\
&= \varphi(z) + \int_{\gamma_{\widetilde{r}}} -G_{\widetilde{r}}(z,w)\mu(dw) - H(z) - \int_{\gamma_{\widetilde{r}}} -G_{\widetilde{r}}(z,w)hn_u(dw)\\
&= \varphi(z) + R(z),
\end{aligned}$$

where the last equality defines $R(z)$.

Inserting this in (11.27), we get

$$\int \chi(z)hn_u(dz) = \int \chi(z)\mu(dz) + \int R(z)\Delta\chi(z)L(dz). \qquad (11.29)$$

(Here we also used some extension of φ to Γ with $\mu = \Delta\varphi$.) The task is now to estimate $R(z)$ and the corresponding integral in (11.29). Put

$$M_j = \mu(\Omega_j),\ \Omega_j = D(z_j, C_1 r)\cap\gamma_r. \qquad (11.30)$$

Using the exponential decay property (11.7) (equally valid for $G_{\widetilde{r}}$) we get for $z\in\Omega_j\cap\gamma_{\widetilde{r}}$, $\mathrm{dist}\,(z,\partial(D(z_j,C_1 r)\cap\gamma_{\widetilde{r}}))\ge r/\mathcal{O}(1)$:

$$\int_{\gamma_{\widetilde{r}}} -G_{\widetilde{r}}(z,w)\mu(dw) \le \int_{\Omega_j\cap\gamma_{\widetilde{r}}} -G_{\widetilde{r}}(z,w)\mu(dw) + \mathcal{O}(1)\sum_{k\ne j} M_k e^{-\frac{1}{C_0}|j-k|}. \qquad (11.31)$$

Similarly from (11.24), we get

$$H(z) \le \mathcal{O}(1)(\epsilon + \int_{\Omega_j\cap\gamma_{\widetilde{r}}} -G_{\widetilde{r}}(z_j,w)\mu(dw) + \sum_{k\ne j} e^{-\frac{1}{C_0}|j-k|}M_k), \qquad (11.32)$$

for $z\in\Omega_j\cap\gamma_{\widetilde{r}}$.

This gives the following estimate on the contribution from the first two terms in $R(z)$ to the last integral in (11.29):

$$\begin{aligned}
&\int_{\gamma_{\widetilde{r}}}(\int_{\gamma_{\widetilde{r}}} -G_{\widetilde{r}}(z,w)\mu(dw) - H(z))\Delta\chi(z)L(dz) \qquad (11.33)\\
&= \mathcal{O}(1)(N\epsilon + \sum_j(\sup_{z\in\Omega_j\cap\gamma_{\widetilde{r}}}\int_{\Omega_j\cap\gamma_{\widetilde{r}}} -G_{\widetilde{r}}(z,w)\mu(dw) + \sum_{k\ne j} e^{-\frac{1}{C_0}|j-k|}M_k))\\
&= \mathcal{O}(1)(N\epsilon + \sum_j \sup_{z\in\Omega_j\cap\gamma_{\widetilde{r}}}\int_{\Omega_j\cap\gamma_{\widetilde{r}}} -G_{\widetilde{r}}(z,w)\mu(dw) + \mu(\gamma_r)).
\end{aligned}$$

The contribution from the last term in $R(z)$ (in (11.28)) to the last integral in (11.29) is

$$\int_{z\in\gamma_{\widetilde{r}}}\int_{w\in\gamma_{\widetilde{r}}} G_{\widetilde{r}}(z,w)hn_u(dw)\Delta\chi(z)L(dz). \tag{11.34}$$

Here

$$\begin{aligned} &\int_{z\in\gamma_{\widetilde{r}}} G_{\widetilde{r}}(z,w)(\Delta\chi)(z)L(dz) \\ =\ &\int_{\widetilde{z}\in\widetilde{r}^{-1}\gamma_{\widetilde{r}}} G_{\widetilde{r}}(\widetilde{r}\widetilde{z},\widetilde{r}\widetilde{w})\Delta_z\chi(\widetilde{r}\widetilde{z})\widetilde{r}^2L(d\widetilde{z}) \\ =\ &\int G_{\widetilde{r}^{-1}\gamma_{\widetilde{r}}}(\widetilde{z},\widetilde{w})\Delta_{\widetilde{z}}(\chi(\widetilde{r}\widetilde{z}))L(dz)=\mathcal{O}(1), \end{aligned}$$

so the expression (11.34) is

$$\begin{aligned} &\mathcal{O}(h)\#(u^{-1}(0)\cap\gamma_{\widetilde{r}}) \\ =\ &\mathcal{O}(1)(\frac{\epsilon}{r}+\sum_{j=1}^{N}\int_{\gamma_r}(-G_r(z_j,w))\mu(dw)) \\ =\ &\mathcal{O}(1)(\frac{\epsilon}{r}+\sum_{j=1}^{N}\int_{\Omega_j}-G_r(z_j,w)\mu(dw)+\mu(\gamma_r)). \end{aligned} \tag{11.35}$$

Using all this in (11.29), we get

$$\begin{aligned} &\int\chi(z)hn_u(dz)=\int\chi(z)\mu(dz) \\ &+\mathcal{O}(1)(\frac{\epsilon}{r}+\sum_j(\sup_{z\in\Omega_j\cap\gamma_{\widetilde{r}}}\int_{\Omega_j\cap\gamma_{\widetilde{r}}}-G_{\widetilde{r}}(z,w)\mu(dw)+\int_{\overline{\Omega_j}}-G_r(z_j,w)\mu(dw))+\mu(\gamma_r)). \end{aligned} \tag{11.36}$$

We replace the smoothness assumption on φ by the assumption that φ is continuous near Γ and keep (11.14). Then by regularization, we still get (11.36).

In order to simplify this further, we introduce a weak regularity assumption on the measure μ. Assume first that $\mu=\Delta\varphi$ is defined in a fixed r-independent neighborhood of $\partial\Gamma$. For $D(z,t)$ contained in that neighborhood we assume that as $t\to 0$,

$$W_z(t):=\mu(D(z,t))=\mathcal{O}(t^{\rho_0}), \tag{11.37}$$

for some $0<\rho_0\le 2$.

Remark 11.2 It is easy to see that this assumption on $\Delta\varphi$ implies that φ is continuous near Γ. In the case $\rho_0 > 1$, we notice that as $r \to 0$,

$$\mu(\gamma_r) = \mathcal{O}(r^{\rho_0 - 1}). \tag{11.38}$$

(This is true also for $\rho_0 \leq 1$ but then of no interest.)

Lemma 11.3 *Assume (11.37) for some $\rho_0 \in]0, 2]$. Then for every domain $\Omega \subset \gamma_r$ and every $z \in \Omega \cap \gamma_{(1-\frac{1}{C})r}$, we have for $0 < t \leq r/2$:*

$$\int_\Omega -G_r(z,w)\mu(dw) \leq \mathcal{O}(1) t^{\rho_0} \ln \frac{r}{t} + \mathcal{O}(1) \ln(\frac{r}{t})\mu(\Omega). \tag{11.39}$$

Proof: Write

$$\int_\Omega -G_r(z,w)\mu(dw) = \int_{D(z,t)\cap\Omega} -G_r(z,w)\mu(dw) + \int_{\Omega\setminus D(z,t)} -G_r(z,w)\mu(dw).$$

For $|z-w| \geq t$, we have $-G_r(z,w) \leq \mathcal{O}(1) \ln \frac{r}{t}$ (cf (11.5)), so the last integral is $\mathcal{O}(1)\ln(\frac{r}{t})\mu(\Omega)$. For $w \in D(z,t) \cap \Omega$, we have

$$-G_r(z,w) \leq \mathcal{O}(1) \ln \frac{r}{|z-w|},$$

hence

$$\begin{aligned}\int_{D(z,t)\cap\Omega} -G_r(z,w)\mu(dw) &\leq \mathcal{O}(1) \int_0^t \ln \frac{r}{s} dW_z(s) \\ &= \mathcal{O}(1)([\ln(\frac{r}{s})W_z(s)]_0^t + \int_0^t \frac{1}{s} W_z(s) ds) \\ &= \mathcal{O}(1) t^{\rho_0} \ln \frac{r}{t}.\end{aligned}$$

$\square$

Corollary 11.4 *Under the same assumptions, we have for every $N \in \mathbf{N}$:*

$$\int_\Omega -G_r(z,w)\mu(dw) \leq \mathcal{O}_N(1)(r^N + \ln(\frac{1}{r})\mu(\Omega)). \tag{11.40}$$

Proof: We just choose $t = r^M$, $0 < M \in \mathbf{N}$ and use that $\ln r^{-M} = M \ln r^{-1}$ $\square$

If we assume (11.38), then the corollary allows us to simplify (11.36) to

$$\int \chi(z) h n_u(dz) = \int \chi(z)\mu(dz) + \mathcal{O}(1)\frac{\epsilon}{r} + \mathcal{O}_N(r^N + \ln(\frac{1}{r})\mu(\gamma_r)). \tag{11.41}$$

Summing up the discussion, we have proved

Proposition 11.5 *Let $\Gamma \Subset \mathbf{C}$ have smooth boundary and let φ be a continuous subharmonic function defined near $\overline{\Gamma}$. Then we have the following result, valid uniformly for $0 < \epsilon \ll 1$, $0 < r \ll 1$, $0 < h \ll 1$: Let u be a holomorphic function, defined in $\Gamma + D(0, r)$ with $h \ln |u(z)| \le \varphi(z)$, $z \in \partial\Gamma + D(0, r)$ and assume that there exist $z_1, ..., z_N \in \partial\Gamma + D(0, \frac{r}{2})$ such that*

$$\partial\Gamma + D(0, r) \subset \bigcup_1^N D(z_j, 2r),\ N \backsim \frac{1}{r},\ h \ln |u(z_j)| \ge \varphi(z_j) - \epsilon. \tag{11.42}$$

Then with $\Omega_j = D(z_j, 2r) \cap (\partial\Gamma + D(0, r))$, we have

$$\begin{aligned} &|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{2\pi h}\int_\Gamma \Delta\varphi L(dz)| \le \frac{\mathcal{O}(1)}{h}\Big(\frac{\epsilon}{r} + \mu(\gamma_r) \\ &+ \sum_j \big(\sup_{z \in \Omega_j \cap (\partial\Gamma + D(0, \frac{r}{4}))} \int_{\Omega_j \cap (\partial\Gamma + D(0, \frac{r}{2}))} -G_{\frac{r}{2}}(z, w)\mu(dw) + \int_{\overline{\Omega}_j} -G_r(z_j, w)\mu(dw)\Big). \end{aligned} \tag{11.43}$$

If we assume also that (11.37) holds for some $0 < \rho_0 \le 2$, then we have for every $N > 0$:

$$\begin{aligned} &|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{2\pi h}\int_\Gamma \Delta\varphi L(dz)| \le \\ &\frac{\mathcal{O}(1)}{h}(\frac{\epsilon}{r} + \mathcal{O}_N(1)(r^N + \ln(\frac{1}{r})\mu(\partial\Gamma + D(0, r)))). \end{aligned} \tag{11.44}$$

Example 11.6 If φ is of class C^2 near the boundary, then (11.37) is satisfied with $\rho_0 = 2$ and $\mu(\partial\Gamma + D(0, r)) = \mathcal{O}(r)$. We choose $N = 1$ so that the right hand side of (11.44) becomes

$$\frac{\mathcal{O}(1)}{h}(\frac{\epsilon}{r} + r \ln \frac{1}{r}).$$

If we choose $r = \sqrt{\epsilon}$, we get

$$|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{2\pi h}\int_\Gamma \Delta\varphi(z) L(dz)| \le \frac{\mathcal{O}(1)}{h}\sqrt{\epsilon}\ln\frac{1}{\epsilon}.$$

In this case we loose a factor $\ln \epsilon^{-1}$ compared to Proposition 6.1 in [7].

12 Spectral asymptotics in a more general case

Let $\Gamma \Subset \widetilde{\Omega}$ be open with C^∞ boundary. For z in a neighborhood of $\partial\Gamma$ and $0 < s, t \ll 1$, we put

$$V_z(t) = \mathrm{Vol}\,\{\rho \in \mathbf{R}^{2n};\ |p(\rho) - z|^2 \le t\},\ W_z(s) = V_z(s^2). \tag{12.1}$$

Recall that in any bounded domain in phase space, the symbols $|p_z(\rho)|^2 = q_z(\rho)$ and $|p(\rho) - z|^2$ are uniformly of the same order of magnitude. If we replace $|p(\rho) - z|^2$ by $q_z(\rho)$ in (12.1), we get a new function $V_z^{\mathrm{new}}(t)$ such that

$$V_z(\frac{t}{C}) \le V_z^{\mathrm{new}}(t) \le V_z(Ct). \tag{12.2}$$

For the purposes of this paper, we can therefore identify the two functions and resort to the second definition whenever we find it convenient. Also, when z is fixed, we will sometimes write $V(t)$ instead of $V_z(t)$.

Our weak assumption, replacing (10.1), is

$$\exists \kappa \in]0,1],\ \text{such that } V_z(t) = \mathcal{O}(t^\kappa),\ \text{uniformly for } z \in \mathrm{neigh}\,(\partial\Gamma),\ 0 \le t \ll 1. \tag{12.3}$$

Example 12.1 When (10.1) holds for $z \in \mathrm{neigh}\,(\partial\Gamma)$, it is clear that (12.3) is fulfilled with $\kappa = 1$ and in particular this is the case when $\{p, \overline{p}\} \neq 0$ on $p^{-1}(\partial\Gamma)$. If $z \in \partial\Sigma \setminus \Sigma_\infty$ then (10.1) cannot hold, so if $\partial\Gamma \cap \partial\Sigma \neq 0$, the best we can hope for is that

$$\forall z \in p^{-1}(\partial\Gamma),\ \text{either } \{p, \overline{p}\} \neq 0 \text{ or } \{p, \{p, \overline{p}\}\} \neq 0. \tag{12.4}$$

This is the situation considered in the 1-dimensional case in [8] where deterministic upper bounds on the density of the eigenvalues were obtained. Following some arguments there, we shall see that *if (12.4) holds, then (12.3) holds with* $\kappa = \frac{3}{4}$. In fact, if we assume (12.4) and if $p(\rho_0) = z_0 \in \partial\Gamma$, we estimate the contribution to $W_{z_0}(\tau)$ from a neighborhood of ρ_0 in the following way:

If $|\{p, \overline{p}\}(\rho_0)| \ge 1/C$, then $d\mathrm{Re}\,p, d\,\mathrm{Im}\,p$ are independent near ρ_0 and the contribution is $\mathcal{O}(\tau^2)$. If $|\{p, \overline{p}\}(\rho_0)|$ is very small, we know that $|\{p, \{p, \overline{p}\}\}(\rho_0)| \ge 1/C$ and in order to fix the ideas we assume that $H_{\mathrm{Re}\,p}^2 \,\mathrm{Im}\,p(\rho_0) \ge 1/C$. This means that

$$H := \{\rho;\ \{\mathrm{Re}\,p, \mathrm{Im}\,p\}(\rho) = 0\}$$

is a smooth hypersurface in a neighborhood of ρ_0 and that $H_{\mathrm{Re}\,p}$ is transversal to H there. A general point in a neighborhood of ρ_0 can therefore we written

$$\rho = \exp{(tH_{\mathrm{Re}\,p})}(\rho'),\ t \in \mathrm{neigh}\,(0, \mathbf{R}),\ \rho' \in H.$$

Then $\operatorname{Re} p(\rho) = \operatorname{Re} p(\rho')$, $\operatorname{Im} p(\rho) = \operatorname{Im} p(\rho') + t^2 g(t,\rho)$, $g > 1/C$. Write $\operatorname{Re} p = s$ so that a general point ρ' in H is parametrized by (s, ρ'') with $\rho'' \in \operatorname{neigh}(0, \mathbf{R}^{2n-2})$.

Write $z_0 = x_0 + iy_0$. For every fixed ρ'', if $|p(\rho) - z_0| \leq \tau$ then $|s - x_0| \leq \tau$ and $|\operatorname{Im} p(\rho') + t^2 g(t, s, \rho'') - y_0| \leq \tau$. Then we are confined to an interval of length 2τ in the s-variable and, for every such fixed s, to an interval of length $\mathcal{O}(\tau^{1/2})$ in the t-variable, or to the union of two such intervals. By Fubini's theorem, the contribution to the volume is therefore $\mathcal{O}(\tau^{3/2})$. Hence $W_{z_0}(\tau) = \mathcal{O}(\tau^{3/2})$, so $V_{z_0}(t) = \mathcal{O}(t^{3/4})$, as claimed.

In Section 10 we introduced the distribution $I(z)$ in (10.2) and showed that $I(z)$ is subharmonic, satisfying (10.3). This implies that

$$\int_{D(z,s)} \Delta(I(w))L(dw) = 2\pi W_z(s), \tag{12.5}$$

and (12.3) is equivalent to

$$\int_{D(z,t)} \Delta(I(w))L(dw) = \mathcal{O}(t^{\rho_0}), \text{ uniformly for } z \in \operatorname{neigh}(\partial\Gamma),\ 0 \leq t \ll 1, \tag{12.6}$$

with $\rho_0 = 2\kappa \in]0, 2]$. This is precisely the condition (11.37) for $I = \varphi$, $\mu = \Delta I$. In view of (12.2), the assumption (12.3) is also equivalent to requiring (4.21) to hold uniformly for $z \in \operatorname{neigh}(\partial\Gamma)$ with $q = q_z = |(\widetilde{p} - z)^{-1}(p - z)|^2$.

Consider the holomorphic function

$$F_\delta(z; h) = \det P_\delta(z),\ z \in \widetilde{\Omega}, \tag{12.7}$$

where we recall that $P_\delta(z) = (\widetilde{P} - z)^{-1}(P_\delta - z)$. Theorem 9.3 and its proof give:

Proposition 12.2 *Let δ satisfy (9.17). Then there exist constants $C, C_0, \widetilde{C} > 0$ such that*
(a) With probability $\geq 1 - Ce^{-C_0 h^{-2n}}$, we have

$$\ln |F_\delta(z; h)| \leq \frac{1}{(2\pi h)^n}(I(z) + Ch^{\delta_0} \ln \frac{1}{h}), \tag{12.8}$$

for all z in some fixed neighborhood of $\partial\Gamma$.
(b) For every $z \in \operatorname{neigh}(\partial\Gamma)$, $\epsilon \geq 0$, we have

$$\ln |F_\delta(z; h)| \geq \frac{1}{(2\pi h)^n}(I(z) - Ch^{\delta_0}(\ln \frac{1}{h} + \ln \frac{1}{\delta}) - \epsilon), \tag{12.9}$$

with probability $\geq 1 - Ce^{-\epsilon(2\pi h)^{-n}} - \widetilde{C}e^{-C_0 h^{-2n}}$.

For the upper bound (12.8), we recall that the upper bound (8.18) was obtained when $\|Q\|_{\mathrm{HS}}$ satisfies the estimate (8.9) and this event is independent of z.

We can now apply Proposition 11.5, with φ equal to $I + Ch^{\kappa}\ln\frac{1}{h}$ and with h there replaced by $(2\pi h)^n$, with ϵ in (11.14) replaced by

$$\mathcal{O}(1)(h^{\kappa}\ln\frac{1}{h} + h^{\kappa}\ln\frac{1}{\delta} + \epsilon),$$

and with ϵ in (12.9) large enough, so that ϵ is the dominant term in the last expression. In other words, we take

$$\epsilon \gg h^{\kappa}\ln\frac{1}{\delta}, \tag{12.10}$$

using also that $\ln\delta^{-1} \geq \ln h^{-1}$.

For $0 < r \ll 1$, choose $z_1, ..., z_N$ and N as in the first part of (11.42). Then in view of (b) in the proposition, the last estimate in (11.42) (with h there replaced by $(2\pi h)^n$) holds for all j with a probability

$$\geq 1 - \frac{C}{r}e^{-\frac{\epsilon}{2}(2\pi h)^{-n}} - \widetilde{C}e^{-C_0 h^{-2n}}.$$

The term

$$\frac{1}{2\pi h}\int_{\Gamma}\Delta\varphi L(dz)$$

in (11.44) becomes after the substitutions $h \mapsto (2\pi h)^n$, $\varphi \mapsto I$:

$$\frac{1}{(2\pi h)^n}\int_{\Gamma}\frac{\Delta I(z)}{2\pi}L(dz) = \frac{1}{(2\pi h)^n}\mathrm{Vol}\,(p^{-1}(\Gamma)),$$

where we also used (10.3).

Theorem 12.3 *Let δ satisfy (9.17). Assume (12.3), with $\kappa \in]0,1]$. Let $N(P + \delta Q_{\omega}, \Gamma)$ be the number of eigenvalues of $P + \delta Q_{\omega}$ in Γ. Then for every fixed $K > 0$ and for $0 < r \ll 1$:*

$$\begin{aligned}&|N(P+\delta Q_{\omega},\Gamma) - \frac{1}{(2\pi h)^n}\iint_{p^{-1}(\Gamma)} dxd\xi| \leq \\ &\frac{C}{h^n}(\frac{\epsilon}{r} + C_K(r^K + \ln(\frac{1}{r})\iint_{p^{-1}(\partial\Gamma + D(0,r))} dxd\xi)),\ 0 < r \ll 1,\end{aligned} \tag{12.11}$$

with probability

$$\geq 1 - \frac{C}{r}e^{-\frac{\epsilon}{2}(2\pi h)^{-n}} \tag{12.12}$$

provided that

$$h^\kappa \ln \frac{1}{\delta} \ll \epsilon \ll 1, \tag{12.13}$$

or equivalently,

$$e^{-\frac{\epsilon}{Ch^\kappa}} \le \delta,\ C \gg 1,\ \epsilon \ll 1,$$

implying that $\epsilon \ge \widetilde{C} h^\kappa \ln \frac{1}{h}$, *for some* $\widetilde{C} > 0$.

In (12.11) we want the right hand side to be much smaller than h^{-n} so it is natural to assume that

$$\ln(\frac{1}{r}) \iint_{p^{-1}(\partial\Gamma + D(0,r))} dx d\xi = \mathcal{O}(r^{\alpha_0}),\ r \to 0, \tag{12.14}$$

for some $\alpha_0 > 0$. When $\kappa \in]\frac{1}{2}, 1]$, we automatically have (12.14) with any $\alpha_0 \in]0, 2\kappa - 1[$. In the right hand side of (12.11), we first choose $N \ge \alpha_0$ and we choose $r = \epsilon^{1/(1+\alpha_0)}$, so that $\epsilon/r,\ r^{\alpha_0} = \mathcal{O}(\epsilon^{\frac{\alpha_0}{1+\alpha_0}})$. Then the right hand side of (12.11) becomes

$$\le \frac{C}{h^n} \epsilon^{\frac{\alpha_0}{1+\alpha_0}}.$$

So, if $1 \gg \epsilon \ge \widetilde{C} h^\kappa \ln \frac{1}{h}$ with $\widetilde{C}$ sufficiently large, and δ is as in the theorem, then

$$|N(P + \delta Q_\omega, \Gamma) - \frac{1}{(2\pi h)^n} \iint_{p^{-1}(\Gamma)} dx d\xi| \le \frac{C}{h^n} \epsilon^{\frac{\alpha_0}{1+\alpha_0}}, \tag{12.15}$$

with probability

$$\ge 1 - \frac{C}{\epsilon^{\frac{1}{1+\alpha_0}}} e^{-\frac{\epsilon}{2}(2\pi h)^{-n}}. \tag{12.16}$$

This expression is very close to 1 except possibly in the case $\kappa = 1$, $n = 1$. In that case, we replace κ by a strictly smaller value and choose δ, ϵ as above.

Theorem 12.4 *Let $\mathcal{G}$ be a family of domains Γ as in Theorem 12.3 satifying the assumptions there uniformly (cf Theorem 10.3) and in particular we assume (12.3) uniformly for all z in a neighborhood of the union of all the $\partial\Gamma$. Then we have (12.11) with probability*

$$\ge 1 - \frac{C}{r^2} e^{-\frac{\epsilon}{(2\pi h)^n}}$$

provided that

$$h^\kappa \ln \frac{1}{\delta} \ll \epsilon \ll 1.$$

13 Appendix: Gaussian random variables in Hilbert spaces

In this appendix we review some generalities about Gaussian random variables in Hilbert spaces that seem to be quite standard to probabilists.

Let $\alpha_1, \alpha_2, ...$ be a sequence of independent $\mathcal{N}(0,1)$-laws, and let $\mathcal{H}$ be a complex separable Hilbert space.

Proposition 13.1 *Let $v_1, v_2, ... \in \mathcal{H}$ be a sequence of vectors such that $\sum_1^\infty \|v_j\|^2 < \infty$, then if the sequence $n_1 < n_2 < ...$ tends to ∞ sufficiently fast, we have that*

$$\lim_{k\to\infty} \sum_1^{n_k} \alpha_j(\omega) v_j \textit{ exists almost surely (a.s.).}$$

Let $S(\omega)$ denote the almost sure limit. If $\widetilde{n}_k$ is another increasing sequence tending to infinity, such that the limit

$$\lim_{k\to\infty} \sum_1^{\widetilde{n}_k} \alpha_j(\omega) v_j =: \widetilde{S}(\omega)$$

exists almost surely, then $\widetilde{S}(\omega) = S(\omega)$ a.s.

Proof: Let (Ω, P) be the underlying probability space. Then $f_j := \alpha_j(\omega) v_j$ can be viewed as elements of $L^2(\Omega, \mathcal{H})$ of norm $\|v_j\|$. They are mutually orthogonal since the α_j are independent. We thus have an orthogonal sum $\sum_1^\infty \alpha_j v_j$ which converges in $L^2(\Omega; \mathcal{H})$ and as usual, using the Chebyschev inequality, we deduce the existence of a sequence of partial sums that converges a.s. $\square$

Let $e_1, e_2, ..$ and $f_1, f_2, ...$ be two orthonormal bases in $\mathcal{H}$. Let $\alpha_1(\omega), \alpha_2(\omega), ...$ be independent complex $\mathcal{N}(0,1)$-laws, and consider the formal vector $\sum_1^\infty \alpha_j(\omega) e_j$. Almost surely, $\{\alpha_j(\omega)\}_1^\infty$ is not in ℓ^2 so our vector is not in $\mathcal{H}$. However, if $v \in \mathcal{H}$, then a.s., we can define the scalar product

$$(\sum_1^\infty \alpha_j(\omega) e_j | v) = \sum_1^\infty \alpha_j(\omega)(e_j|v) \tag{13.1}$$

as in Proposition 13.1, since $\{(e_j|v)\}_1^\infty \in \ell^2$.

We now look for random variables $\beta_1(\omega), \beta_2(\omega), ...$ such that

$$\sum_1^\infty \alpha_k(\omega) e_k = \sum_1^\infty \beta_j(\omega) f_j, \tag{13.2}$$

in the sense that the formal scalar products with $f_1, f_2, ...$ are equal. This leads to the definition

$$\beta_j(\omega) = \sum_{k=1}^{\infty} (e_k|f_j)\alpha_k(\omega), \tag{13.3}$$

which is well-defined as in Proposition 13.1, since $k \mapsto (e_k|f_j)$ is in ℓ^2. For every finite N, the variable

$$\sum_{k=1}^{N} (e_k|f_j)\alpha_k(\omega) \tag{13.4}$$

has the density

$$*_{k=1}^{N} \frac{1}{\pi|(e_k|f_j)|^2} e^{-|\alpha|^2/|(e_k|f_j)|^2},$$

where $*$ indicates convolution products. Hence the characteristic function (i.e. the Fourier transform) is

$$\exp\Big(-\frac{1}{4}(\sum_{k=1}^{N} |(e_k|f_j)|^2)|\xi|^2\Big),$$

so (13.4) is a normal distribution $\mathcal{N}(0, \sum_{k=1}^{N} |(e_k|f_j)|^2)$. The unitarity of the matrix $((e_k|f_j))$ then implies that $\beta_j(\omega)$ is a $\mathcal{N}(0,1)$-law.

Proposition 13.2 *β_j are independent $\mathcal{N}(0,1)$-laws.*

Proof: We have already seen that β_j are $\mathcal{N}(0,1)$-laws. To see that they are independent, we compute (using Proposition 13.1) the joint distribution of $\beta_1, \beta_2, ..., \beta_N$. Write

$$\beta^{(N)} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ .. \\ \beta_N \end{pmatrix} = \sum_{1}^{\infty} \alpha_k \nu_k,$$

where

$$\nu_k = \begin{pmatrix} \sigma_{1,k} \\ \sigma_{2,k} \\ \cdots \\ \sigma_{N,k} \end{pmatrix}, \ \sigma_{j,k} = (e_k|f_j).$$

$\alpha_k(\omega)\nu_k$ is a random variable with values in $\mathbf{C}^N$ and with the characteristic function

$$\begin{aligned}\chi_{\alpha_k\nu_k}(\xi) &= \int e^{-i\mathrm{Re}\,\alpha(\nu_k|\xi)}e^{-|\alpha|^2}\frac{L(d\alpha)}{\pi}\\ &= \exp\left(-\frac{1}{4}|(\nu_k|\xi)|^2\right)\\ &= \exp\left(-\frac{1}{4}\sum_{\ell=1}^N\sum_{m=1}^N \sigma_{\ell,k}\overline{\sigma_{m,k}}\overline{\xi}_\ell\xi_m\right).\end{aligned}$$

It follows that

$$\chi_{\sum_1^\infty \alpha_k\nu_k}(\xi) = \exp\left(-\frac{1}{4}\sum_{\ell=1}^N\sum_{m=1}^N(\sigma_\ell|\sigma_m)\overline{\xi}_\ell\xi_m\right),$$

where $\sigma_j = (\sigma_{j,k})_{k=1}^\infty \in \ell^2$. But the σ_j form an orthonormal system, so finally,

$$\chi_{\sum_1^\infty \alpha_k\nu_k}(\xi) = \exp\, -\frac{1}{4}|\xi|^2.$$

This means that the joint distribution of $\beta_1, ..., \beta_N$ is

$$\frac{1}{\pi^N}e^{-|\beta|^2}L_{\mathbf{C}^N}(d\beta),$$

and that $\beta_1, ..., \beta_N$ are independent. $\square$

The random variable (13.1) is an $\mathcal{N}(0, \|v\|^2)$-law.

If $v \in \mathcal{H}$ is any finite linear combination of the f_j, we know by construction that

$$(\sum_1^\infty \alpha_j(\omega)e_j|v) = (\sum_1^\infty \beta_j(\omega)f_j|v), \text{ a.s.}$$

If $v \in \mathcal{H}$ is arbitrary, we write $v = v_\epsilon + r_\epsilon$, where v_ϵ is a finite linear combination of the f_j and $\|r_\epsilon\| < \epsilon$. We conclude that almost surely,

$$(\sum_1^\infty \alpha_j(\omega)e_j|v) = (\sum_1^\infty \beta_j(\omega)f_j|v) + (\sum_1^\infty \alpha_j(\omega)e_j|r_\epsilon) - (\sum_1^\infty \beta_j(\omega)f_j|r_\epsilon).$$

Here the last two terms are $\mathcal{N}(0, \|r_\epsilon\|^2)$-laws and hence as small as we like with a probability as close as we like to 1, when ϵ is small eneough. We conclude that

$$(\sum_1^\infty \alpha_j(\omega)e_j|v) = (\sum_1^\infty \beta_j(\omega)f_j|v) \text{ a.s.} \tag{13.5}$$

Proposition 13.3 *Let $\mathcal{H}$, $\widetilde{\mathcal{H}}$ be two separable Hilbert spaces and let $T : \mathcal{H} \to \widetilde{\mathcal{H}}$ be a Hilbert-Schmidt operator. Let $\alpha_j(\omega)e_j$, $\beta_j(\omega)f_j$ be as above. Then $T(\sum_1^\infty \alpha_j(\omega)e_j)$ is well defined a.s. and equal to $T(\sum_1^\infty \beta_j(\omega)f_j)$ a.s.*

Proof: We define $T(\sum_1^\infty \alpha_j(\omega)e_j)$ as $\sum_1^\infty \alpha_j(\omega)Te_j$ in the sense of Proposition 13.1, using that

$$\sum \|Te_j\|_{\widetilde{\mathcal{H}}}^2 = \|T\|_{\mathrm{HS}}^2 < \infty.$$

Notice also that for every $v \in \widetilde{\mathcal{H}}$ we have a.s.

$$\begin{aligned}(T(\sum_1^\infty \alpha_j(\omega)e_j)|v) &= \sum_1^\infty \alpha_j(\omega)(Te_j|v) \text{ a.s.} \\ &= \sum_1^\infty \alpha_j(\omega)(e_j|T^*v).\end{aligned}$$

The same considerations apply to $T(\sum_1^\infty \beta_j(\omega)f_j)$ so in view of (13.5), for every $v \in \widetilde{\mathcal{H}}$ we have

$$(T(\sum_1^\infty \alpha_j(\omega)e_j)|v) = (T(\sum_1^\infty \beta_j(\omega)f_j)|v) \text{ a.s.}$$

We get the same conclusion a.s. simultaneously for all v in any countable set, and letting $v = g_1, g_2, ...$, where g_j form and orthonormal basis in $\widetilde{\mathcal{H}}$, we conclude that

$$T(\sum_1^\infty \alpha_j(\omega)e_j) = T(\sum_1^\infty \beta_j(\omega)f_j) \text{ a.s.}$$

$\square$

Now let $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ be separable Hilbert spaces and let $T : \mathcal{E} \to \mathcal{F}$, $S : \mathcal{G} \to \mathcal{H}$ be Hilbert-Schmidt operators. If $f \in \mathcal{F}$, $g \in \mathcal{G}$, we also denote by g, f the corresponding multiplication operators $\mathbf{C} \ni z \mapsto zg, zf \in \mathcal{G}, \mathcal{F}$, so that $f^*u = (u|f)$. Then $gf^*u = (u|f)g$ defines an operator : $\mathcal{F} \to G$ which has the Hilbert-Schmidt norm $\|g\|\|f\|$. Let f_j, $j = 1, 2, ...$, g_j, $j = 1, 2, ...$ be orthonormal bases in $\mathcal{F}, \mathcal{G}$. Then $\{g_j f_k^*\}_{j,k=1}^\infty$ is an orthonormal basis for the space HS$(\mathcal{F}, \mathcal{G})$of Hilbert-Schmidt operators $\mathcal{F} \to \mathcal{G}$. Now,

$$Sg_j f_k^* T = (Sg_j)(T^* f_k)^*,$$

and

$$\|Sg_j f_k^* T\|_{\mathrm{HS}}^2 = \|Sg_j\|^2 \|T^* f_k\|^2.$$

It follows that

$$\sum_{j,k} \|Sg_j f_k^* T\|_{\mathrm{HS}}^2 = \|S\|_{\mathrm{HS}}^2 \|T\|_{\mathrm{HS}}^2,$$

and we conclude that

$$\mathrm{HS}(\mathcal{F}, \mathcal{G}) \ni A \mapsto SAT \in \mathrm{HS}(\mathcal{E}, \mathcal{H})$$

is a Hilbert-Schmidt operator. The earlier discussion can therefore be applied:

Proposition 13.4 *Let $\alpha_{j,k}(\omega)$ be independent $\mathcal{N}(0,1)$ laws. Then*

$$S \sum_{j,k} \alpha_{j,k}(\omega) g_j f_k^* T = \sum_{j,k} \alpha_{j,k}(\omega) S g_j f_k^* T$$

is almost surely defined as a Hilbert-Schmidt operator. Moreover, if $\widetilde{g}_j$, $\widetilde{f}_k$ are new orthonormal bases in $\mathcal{G}$, $\mathcal{F}$, then there exists a new set of independent $\mathcal{N}(0,1)$-laws $\beta_{j,k}(\omega)$ such that

$$S \circ (\sum_{j,k} \alpha_{j,k}(\omega) g_j f_k^*) \circ T = S \circ (\sum_{j,k} \beta_{j,k}(\omega) \widetilde{g}_j \widetilde{f}_k^*) \circ T \text{ a.s.} \tag{13.6}$$

References

[1] E.B. Davies, *Semi-classical analysis and pseudospectra,* J. Diff. Eq. 216(1)(2005), 153–187.

[2] M. Dimassi, J. Sjöstrand, *Spectral asymptotics in the semi-classical limit,* London Math. Soc. Lecture Note Series, 268(1999).

[3] E.M. Dynkin, *An operator calculus based upon the Cauchy-Green formula,* J. Soviet Math., 4(4)(1975), 329–334.

[4] M. Embree, L.N. Trefethen, *Spectra and pseudospectra, the behaviour of non-normal matrices and operators,* Princeton Univ. Press, Princeton NJ (2005).

[5] V.I. Girko, *Theory of random determinants,* Mathematics and its applications, Kluwer academic publishers, Dordrecht 1990.

[6] M. Hager, *Instabilité spectrale semiclassique pour des opérateurs non-autoadjoints I: un modèle,* préprint 2004, http://hal.ccsd.cnrs.fr/ccsd-00001594, Annales de la Faculté des Sciences de Toulouse, to appear,

[7] M. Hager, *Instabilité spectrale semiclassique d'opérateurs non-autoadjoints II:* http://hal.ccsd.cnrs.fr/ccsd-00004677.

[8] M. Hager, Thesis (2005), Chapter 3,
http://tel.ccsd.cnrs.fr/docs/00/04/87/08/PDF/tel-00010848.pdf

[9] B. Helffer, J. Sjöstrand, *Équation de Schrödinger avec champs magnétique et équation de Harper,* Springer Lect. Notes in Phys., 345(1989), 118–197.

[10] A. Melin, J. Sjöstrand, *Determinants of pseudodifferential operators and complex deformations of phase space,* Methods Appl. Anal. 9(2)(2002), 177–237.

[11] J. Sjöstrand, *Parametrices for pseudodifferential operators with multiple characteristics,* Ark. f. Mat., 12(1)(1974), 85–130.

[12] J. Sjöstrand, M. Zworski, *Elementary linear algebra for advanced spectral problems,* http://arxiv.org/math.SP/0312166.

[13] L.N. Trefethen, S.J. Chapman, *Wave packet pseudomodes of twisted Toeplitz matrices,* Comm. Pure Appl. Math. 57(2004), 1233–1264.

[14] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung),* Math. Ann., 71(4)(1912), 441–479.